Intrinsic Discretization for Elliptic Boundary Value Problems

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Corina Simian
aus Rumänien

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Prof. Dr. Stefan A. Sauter (Vorsitz)
Prof. Dr. Michel Chipot

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Abstract

The main goal of this thesis is to develop a general approach for the *derivation* of *intrinsic* conforming and non-conforming finite elements from theoretical principles for the discretization of elliptic partial differential equations. We construct intrinsic conforming and non-conforming piecewise polynomial finite element spaces of any order k for elliptic boundary value problems. The proposed intrinsic FEM is based on a simplicial triangulation. We exemplify our method for Poisson's equation and the pure traction problem of linearized elasticity, but we emphasize that this method is applicable also for more general elliptic equations.

In general intrinsic approaches one computes directly physical quantities which otherwise are obtained by numerical differentiation from the primary unknown of the problem. This is for example the case in the direct computation of the fluxes instead of the potential or in the direct computation of the strain or stress tensor instead of the displacement vector. The intrinsic methods have advantages in practical applications avoiding the loss of accuracy by numerical differentiations.

The change of the primary unknown raises a series of questions related to the possibility of adapting the known results from classical FEM theory to the intrinsic theory, questions that we highlight and explain in the thesis.

In the proposed intrinsic approach we develop local conditions for the approximation of the gradient vector field and of the symmetric gradient matrix field and then construct finite element spaces, i.e., local basis functions, directly from these conditions. In order to incorporate the essential boundary conditions we construct lifting operators as the left inverse of elementwise gradient and symmetric gradient operators. A main characteristic of the method is the decomposition of the finite element spaces in a direct sum of vertex-, edge- and triangle-supported subspaces for which basis functions are obtained.

We derive weak continuity conditions for the characterization of the admissible energy space. Based on these conditions we derive conforming intrinsic polynomial finite element spaces and show that in the case of Poisson's equation they are the gradients of the well-known Lagrange hp-finite element spaces and in the case of the pure traction problem of linearized elasticity they are the symmetric gradients of these spaces.

In the non-conforming case we employ the stability and convergence theory for non-conforming finite elements based on the second Strang lemma and derive from these principles weak compatibility conditions at the interfaces between elements of the mesh so that the non-conforming perturbation of the original bilinear form can be estimated in a consistent way. We derive all types of piecewise polynomial finite elements that satisfy this condition and also derive a local basis for these spaces. In the case of Poisson's equation the polynomial non-conforming spaces of degree k are spanned by the gradients of standard hp-finite element basis functions enriched by some edge oriented non-conforming basis functions for k even and by some triangle-supported non-conforming basis functions for k odd. As a by-product, this

methodology allows us to recover the well-known non-conforming Crouzeix-Raviart element, the second order non-conforming Fortin-Soulie element, the third order Crouzeix-Falk element, and the family of Gauss-Legendre elements.
To our knowledge the non-conforming intrinsic method was not treated before to this extend.

Zusammenfassung

Das Ziel dieser Arbeit ist die Entwicklung eines allgemeinen Ansatzes zur Herleitung von intrinsischen konformen und nicht-konformen Finiten Elementen zur Diskretisierung von elliptischen partiellen Differentialgleichungen. Wir konstruieren intrinsische konforme und nicht-konforme stückweise polynomiale Finite Elemene Räume beliebiger Ordnung k für elliptische Randwertprobleme. Die vorgeschlagene intrinsische FEM basiert auf einer Triangulierung mit Simplizes. Wir veranschaulichen unsere Methode am Beispiel der Poisson Gleichung und dem reinen Traktionsproblem für linearisierte Elastizitätsgleichungen, es muss jedoch betont werden, dass diese Methode auch auf allgemeinere elliptische Gleichungen angewendet werden kann.
Bei intrinsischen Methoden berechnet man direkt physikalische Grössen, welche sonst durch numerische Differentiation der ursprünglichen Unbekannten des Problems angenähert werden müssen. Dies ist beispielsweise der Fall bei der direkten Berechnung von Flüssen anstatt von Potentialen oder bei der direkten Berechnung von Spannungs- oder Dehntensoren anstatt von Verschiebungsvektoren. Intrinsische Methoden haben in praktischen Anwendungen den Vorteil, dass der Genauigkeitsverlust durch numerische Differentiation umgangen wird.
Durch den Wechsel der Hauptunbekannten stellt sich die Frage, in wie weit bekannte Resultate der klassischen FEM Theorie auf die intrinsische Theorie angewendet werden können. Diese Fragen werden in dieser Arbeit beleuchtet und erklärt.
Wir entwickeln für die vorgeschlagene intrinsische Methode lokale Bedingungen zur Approximation des Gradientenvektorfelds und des symmetrischen Gradientenmatrix Vektorfelds und konstruieren dann Finite Elemente Räume bzw. lokale Basisfunktionen aus diesen Bedingungen. Zur Berücksichtigung der Randbedingungen konstruieren wir Lift-Operatoren welche linksinvers zum elementweisen Gradienten und zum symmetrischen Gradienten sind. Ein Hauptmerkmal der Methode ist eine Zerlegung der Finite-Elemente-Räume in eine direkte Summe von Teilräumen, die mit den Ecken, Kanten und Dreicken der Triangularisierung verknüpft sind. Wir geben für diese Teilräume explizit Basisfunktionen an.
Wir leiten schwache Stetigkeitsbedingungen zur Charakterisierung des zulässigen Energieraums her. Basierend auf diesen Bedingungen leiten wir konforme intrinsische polynomiale Finite-Elemente-Räume her und zeigen, dass diese, im Fall der Poisson Gleichung, die Gradienten der bekannten Lagrange hp-Finite-Elemete-Räume sind und im Fall des reinen Traktionsproblems für linearisierte Elastizitätsgleichungen die symmetrischen Gradienten dieser Räume sind.
Im nicht-konformen Fall benutzen wir die Stabilitäts- und Konvergenztheorie für nicht-konforme Finite Elemente, die auf dem zweiten Strang Lemma basiert und leiten aus diesen Aussagen schwache Kompatibilitätsbedinungen and den Schnittstellen des Gitters her, sodass nicht-konforme Störungen der original Bilinearform abgeschätzt werden können.
Wir konstruieren alle stückweise polynomialen Finite Elemente, welche diese Bedingungen

erfüllen und geben eine lokale Basis dieser Räume an. Im Fall der Poisson-Gleichung werden die polynomialen nicht-konformen Räume vom Grad k durch die Gradienten der gewöhnlichen hp-Finite-Elemente-Basisfunktionen aufgespannt, wobei im Fall, dass k gerade ist, einige nicht-konforme Basisfunktionen bezüglich der Kanten und im Fall, dass k ungerade ist, einige nicht-konforme Basisfunktionen bezüglich der Dreiecke hinzugefügt werden.
Als Nebenprodukt liefert dieser Ansatz die bekannten nicht-konformen Crouzeix-Raviart-Elemente, das nicht-konforme Fortin-Soulie-Element zweiter Ordnung, das Crouzeix-Falk-Element dritter Ordnung und die Familie von Gauss-Legendre-Elementen. Der nicht-konforme intrinsische Ansatz wurde unserer Meinung nach in dieser Breite und Tiefe zuvor nicht behandelt.

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Contents

List of Notations

Notations of spaces

$\mathbb{R}$	The space of real numbers
$C^\infty(\Omega)$	The space of infinitely differentiable functions over Ω
$C_0^\infty(\Omega)$	The space of infinitely differentiable functions with compact support in Ω
$L^2(\Omega)$	The space of square integrable functions over Ω
$H^1(\Omega)$	The usual Sobolev space which contains the $L^2(\Omega)$ functions with weak first derivatives in $L^2(\Omega)$
$H_0^1(\Omega)$	The space which contains the functions in H^1 with zero traces at the boundary Γ of Ω
$H^m(\Omega)$	The usual Sobolev space which contains the $L^2(\Omega)$ functions with weak derivatives of degree α, $\lvert\alpha\rvert \leq m$ in $L^2(\Omega)$
$H_0^m(\Omega)$	The space which contains the functions in H^m with zero traces at the boundary Γ of Ω
$H^{-m}(\Omega)$	The dual space of $H_0^m(\Omega)$
$W^{r,\infty}(\Omega)$	Sobolev space which contains the $L^\infty(\Omega)$ functions with weak derivatives of degree α, $\lvert\alpha\rvert \leq r$ in $L^\infty(\Omega)$
$\mathcal{P}^n$	The space of polynomials of total degree at most n
V/R	The quotient space of a vector space V modulo the subspace $R \subseteq V$
V'	The dual space of V
$\mathbb{D}_s(\Omega)$	The space of all symmetric tensor fields with infinitely differentiable components with compact support in Ω
$\mathbb{S}^d$	The space of all symmetric matrices of dimension d

The corresponding spaces for vector fields are denoted using boldface letters: $\mathbf{L}^2(\Omega)$, $\mathbf{H}^1(\Omega)$, $\mathbf{H}_0^1(\Omega)$, $\mathbf{P}^n$, etc.
The corresponding spaces of matrix and tensor vector fields are denoted using: $\mathbb{L}^2(\Omega)$, $\mathbb{H}^1(\Omega)$, $\mathbb{H}_0^1(\Omega)$, $\mathbb{P}^n$, $\mathbb{S}^d$, etc.
The spaces of symmetric vector and tensor fields are denoted by using a subscript s: $\mathbf{H}_s^1(\Omega)$, $\mathbf{P}_s^n$, $\mathbb{L}_s^2(\Omega)$ etc.

Notations of operators

div	The divergence operator
∇	The gradient operator
∇_s	The symmetric gradient operator
$curl$	The rotational operator (scalar curl)
$\partial_n u$	The normal derivative of a function u
$\partial_t u$	The tangential derivative of a function u
Δ	The Laplace operator

The corresponding operators acting on vector or tensor spaces are denoted using boldface symbols.

Notations of piecewise operators related to a triangulation $\mathcal{T}$

$\nabla_{\mathcal{T}}$	The piecewise *gradient* operator
$curl_{\mathcal{T}}$	The piecewise *curl* operator

Notations related to the triangulation

$\mathcal{T}_h$	A triangulation of Ω
$\hat{\tau}$	The unit triangle
$\hat{\mathcal{N}}^p$	The nodal points of the unit triangle for polynomials of degree p
$\mathcal{E}$	The set of all interior edges
$\mathcal{V}$	The set of interior vertices
$\mathcal{E}_{\partial\Omega}$	The set of edges lying on $\partial\Omega$
$\mathcal{V}_{\partial\Omega}$	The set of vertices lying on $\partial\Omega$
$\mathcal{E}_V$	The set of edges with a common vertex V
$\mathcal{T}_V$	The set of triangles with a common vertex V
$\mathcal{T}_E$	The set of triangles sharing the edge E

Notations of norms

$\|\cdot\|_{0,\Omega}$	The norm in $L^2(\Omega)$ or in $\mathbf{L}^2(\Omega)$, depending on the context
$\|\cdot\|_{m,\Omega}$, $m \in \mathbb{Z}$	The norm in $H^m(\Omega)$ or in $\mathbf{H}^m(\Omega)$, depending on the context
$\|\cdot\|_E$	The norm in a Hilbert space E
$\|\cdot\|_h$	A mesh dependent norm

Other notations

Γ	Boundary of Ω
Ω	Domain in $\mathbb{R}^2$ or $\mathbb{R}^3$ depending on the problem considered
h	The discretization parameter
${}_{V'}\langle\cdot,\cdot\rangle_V$	The duality pairing between the topological space V and its dual V'

Introduction

The main goal of this thesis is to develop a general method for the derivation of intrinsic conforming and non-conforming finite element spaces for the discretization of elliptic partial differential equations. We introduce a new intrinsic perspective for the development of conforming and non-conforming finite element spaces for elliptic boundary value problems. We apply our approach in the cases of Poisson's equation and of the pure traction problem of linearized elasticity, but this method is applicable also for general elliptic equations. The difference of our method from the other existing intrinsic approaches is that it provides explicitly intrinsic conforming and non-conforming piecewise polynomial finite element spaces of any degree $p \in \mathbb{N}$. In the non-conforming case our approach overcomes the difficulties that appear for the classical non-conforming spaces with even degree. The proposed FEM is based on a simplicial triangulation in the sense of [23].

Unlike the usual methods, the intrinsic method aims for a direct computation of fluxes instead of the computation of the potential for problems in potential theory and the direct computation of the linearized strain tensor field instead of finding the displacement vector field in elasticity problems. This has an exceptional benefit in practical applications, where one is interested to compute directly different physical quantities as the flux, the electrostatic field, the velocity field or the strain tensor. These are only few of the examples arising in practical problems, having also the possibility to extend these results to real-life problems from other fields, like in medicine or biology (e.g.: the study of the elasticity of different cells or organs).

The idea of an intrinsic approach goes back to 1941 to the theory of shells and plates, introduced by Synge and Chien [58]. However an intrinsic FEM approach was only introduced in 2005 for linearized elasticity problems, by P.G. Ciarlet and P. Ciarlet Jr. in [24]. The method was rigourously analyzed by the same authors in other papers [25, 26, 27] and extended also to intrinsic shell theory [20]. Recently, study on intrinsic nonlinear elasticity have been undertaken first for thin elastic shells by S.Opoka and W. Pietraszkiewicz and in [50] and then for three dimensional nonlinear elasticity by P. Ciarlet and C. Mardare in [21, 22].

The intrinsic FEM methods starts from the classical FEM theory. We refer here to the reference monograph of P. Ciarlet [23], D. Braess [31], S.C. Brenner and L.R. Scott [12], C. Schwab [53]. Theoretical foundations of shells theory can be found in [16, 39] and of nonlinear theory of elasticity in [49].

The change of the primary unknown raises a series of problems related to the possibility of adapting known results to the intrinsic theory. One of them is to find "Donati-like" characterizations and Saint-Venant conditions capable to assure the reformulation of the problem in an intrinsic way. In [3] it is proved that Saint Venant's theorem is the matrix analogue of Poincaré's lemma. Different extensions of Donati's characterization have been done: such extensions can be found in [59, 60, 45, 3, 4]. In [59] Ting extended Donati's characterization to matrix fields with components in L^2, in [3] Saint Venant's characterization has been extended

likewise and in [4] both Saint Venant's characterization and Donati's characterization were extended to matrix fields whose components are in H^{-1}. An overview of the characterization theorems applied for development of intrinsic approaches in elasticity is given in [19].
Another problem to be solved consists of the error analysis and convergence theory ([26], [24]).
In our intrinsic approach we develop conditions for the approximation of the flux variable and the symmetric gradient matrix field and then construct a finite element space, i.e., the local basis functions, directly from these conditions. In order to take essential boundary conditions into account we have to construct lifting operators as the left inverse of the elementwise gradient and symmetric gradient operators, that is, operators defined element by element. An important part in our intrinsic approach consists in finding such operators which allow us to switch from the classical case to the "intrinsic" one. Using these operators it is possible to reformulate the initial variational formulation for the finite element discretization as a minimization problem or a constrained minimization problem in terms of the fluxes. The conforming and non-conforming finite element spaces can be expressed also using these operators.
We construct conforming and non-conforming intrinsic finite element spaces defining basis functions whose support are given by a single triangle $\tau \in \mathcal{T}$, edge-oriented basis functions whose support are given by two adjacent triangles and vertex-oriented basis functions whose support are given by triangles which share a common vertex. In this way we obtain a decomposition of the finite element space into a direct sum of triangle-, edge- and vertex-oriented subspaces.
The idea of the decomposition of the local and global finite element space into nodal-, edge-, and triangle-oriented local subspaces can be found also in other articles but in a different context. In [52, 61] the global scalar finite element space $W_{p+1} \subset H^1$ and the $H(curl)$ finite element space V_p are split into vertex, edge, and element oriented subspaces. A similar decomposition is made for $H(div)$. The aim of this decomposition is to construct the basis functions such that each one of the blocks satisfies a local complete sequence property. In [36] vertex, edge, and element subspaces are defined using degrees of freedom for each of these elements. The aim is to study if these degrees of freedom are linear independent and which of them must be retained for defining a global basis of the finite element space. In [40] the local and global vertex, edge, and element based subspaces are defined using degrees of freedom associated to these 3 kind of subspaces. Basis functions for these subspaces are not given explicitly.
This thesis consists of 4 chapters. In the first chapter we recall basic principles of the finite element method, including the steps of the FEM discretization and classical results of the convergence theory. We also introduce the model problems, e.g. Poisson's equation and linearized elasticity equations, which will be considered later in Chapters 3 and 4. In Chapter 2 we outline the basic concepts of intrinsic finite element discretizations. We relate the classical unknowns in the problems mentioned above to the corresponding unknowns in the intrinsic approach by different isomorphisms and generalizations of Saint Venant's and Donati's characterizations. In Chapter 3 we apply the proposed intrinsic approach to Poisson's equation with Dirichlet boundary conditions. We construct explicitly conforming and non-conforming finite elements spaces and derive error estimates for the resulting scheme. In Chapter 4 we extend the approach developed in Chapter 3 to linearized elasticity problems and we determine explicitly a conforming finite element space for these problems.

1

Basic Concepts of FEM

In this chapter we present definitions and known results that are needed in order to introduce our intrinsic FEM approaches in Chapter 3 and 4.
Many phenomena are modeled using partial differential equations. The difficulties that arise in finding analytical solutions of these equations lead to the necessity of discretization, finite element method being one of the most powerful and used method for this purpose. FEM is based on the variational formulation of a boundary-value problem. In the following we will briefly introduce the abstract mathematical formulation of the finite element method. The abstract formulation will then be particularized for the case of Poisson's equation and linearized elasticity that we will use in Chapter 3, 4.
A formal definition of finite elements was given by Ciarlet in [23] and it remains the standard definition ([41, 12, 13, 10]) which we also consider in this thesis.
FEM is based on variational principles.

1.1 Basic principles of the FEM discretization

Let Ω be a domain in $\mathbb{R}^d$, let $H^m(\Omega)$ be the usual Sobolev space which contains $L^2(\Omega)$ functions with weak first derivatives of degree α, with $|\alpha| \leq m$ in $L^2(\Omega)$ and $H_0^1(\Omega) \subset H^1(\Omega)$ is the subspace of functions in $H^1(\Omega)$ with zero traces at the boundary Γ of Ω.
A practical problem that can be formulated as boundary-value problem can be solved with the finite element method.
We restrict ourselves here to the case of a linear variational problem in the following form:
Find $u \in V$ such that

$$a(u, v) = L(v), \text{ for all } v \in V, \tag{1.1.1}$$

where V denotes an appropriate Hilbert space ("energy space"), $a : V \times V \to \mathbb{R}$ is a bilinear form and $L : V \to \mathbb{R}$ is a linear form.

Remark 1.1.1. In the case when the bilinear form $a(\cdot, \cdot)$ is symmetric, the variational problem (1.1.1) is equivalent to the minimization problem:
Find $u \in V$ such that

$$J(u) = \inf_{v \in V} J(v), \text{ with } J(v) = \frac{1}{2}a(v, v) - L(v). \tag{1.1.2}$$

Definition 1.1.2. *The bilinear form* $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ *is bounded if there exists a constant* $C \in \mathbb{R},\ C > 0$ *such that:*

$$|a(u, v)| \leq C\|u\|_V \|v\|_V, \text{ for all } u,\, v \in V. \tag{1.1.3}$$

The bilinear form $a(\cdot,\cdot)$ *is coercive if there exists a constant* $\alpha \in \mathbb{R},\ \alpha > 0$ *such that*

$$a(u, u) \geq \alpha\|u\|_V^2, \text{ for all } u \in V. \tag{1.1.4}$$

The existence and uniqueness of the solution of the variational problem is based on the Lax-Milgram Lemma ([23, 44, 6]).

Theorem 1.1.3. *(Lax-Milgram Lemma) Let* V *be a Hilbert space,* $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ *be a bounded, symmetric and coercive bilinear form, then for each* $L \in V'$ *the variational problem defined in (1.1.1) has one and only one solution* $u \in V$ *and*

$$\|u\|_V \leq \frac{1}{\alpha}\|L\|_{V'},$$

where α *is the constant in the coercivity definition of* $a(\cdot,\cdot)$.

The proof can be found in [23]. The theorem implies that the variational problem (1.1.1) is well-posed, i.e., its solution exists, is unique, and depends continuously on L.
A more general form of problem (1.1.1) can be formulate:
Find $u \in V$ *such that*

$$a(u, v) = L(v), \text{ for all } v \in \tilde{V}, \tag{1.1.5}$$

where V and $\tilde{V}$ denotes appropriate Hilbert spaces, $a : V \times \tilde{V} \to \mathbb{R}$ is a bilinear form and $L : \tilde{V} \to \mathbb{R}$ is a linear form.
The existence and uniqueness of this problem is based on a generalized Lax-Milgram Lemma presented in the following theorem:

Theorem 1.1.4. *(Generalized Lax-Milgram Lemma - [6, 44]). Let* V *and* $\tilde{V}$ *be Hilbert spaces and* $a(\cdot,\cdot) : V \times \tilde{V} \to \mathbb{R}$ *be a bounded, symmetric bilinear form satisfying the following two properties:*

1. *There is a constant* $\alpha > 0$ *such that*

$$\inf_{u \in V, \|u\|_V = 1} \ \sup_{v \in \tilde{V}, \|v\|_{\tilde{V}} \leq 1} |a(u, v)| \geq \alpha > 0, \tag{1.1.6}$$

2. $\sup_{u \in V} |a(u, v)| \geq 0, \quad \forall v \in \tilde{V}, v \neq 0.$

Then, for any $L \in \tilde{V}'$ *the variational problem defined in (1.1.5) has one and only one solution* $u \in V$ *and*

$$\|u\|_V \leq \frac{C}{\alpha}\|L\|_{\tilde{V}}',$$

where C *is the constant in the boundedness definition of* a.

Condition (1.1.6) in the generalized Lax-Milgram Lemma is the inf-sup condition of Babuška-Brezzi which generalizes the coercivity condition of the bilinear form a. This condition is the classical condition that needs to be satisfied in order to have a stable and quasi-optimal

procedure for the solution of variational equation (1.1.5).
The discretization of problem (1.1.1) consists in the following two steps:
Step 1 Select a finite dimensional subspace $V_h \subset V$ and solve the discrete problem:
Find $u_h \in V_h$ *such that*

$$a(u_h, v) = L(v), \text{ for all } v \in V_h, \text{ where } V_h \text{ is a subspace of } V. \tag{1.1.7}$$

Existence and uniqueness of the discrete solution is a direct consequence of the Lax-Milgram Lemma given in Theorem 1.1.3 and is formulated in the next theorem:

Theorem 1.1.5. *([44]) Let* $V_h \subseteq V$ *be a subspace of* V*. If the bilinear form* $a : V \times V \to \mathbb{R}$ *is bounded and coercive and* $L \in V'$ *then the discrete variational problem (1.1.7) has an unique solution.*

Remark 1.1.6. If the bilinear form $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ is symmetric, a similar formulation to the one presented in Remark 1.1.1 is possible for the discrete variational problem (1.1.7)*:*
Find $u_h \in V_h$ *such that*

$$J(u_h) = \inf_{v_h \in V_h} J(v_h). \tag{1.1.8}$$

Step 2: Solution of the system of linear equations, obtained by writing the solution u_h in terms of a chosen basis of the finite dimensional space V_h.

The definition of the discrete solution using (1.1.8) is denoted as the Ritz method.
The central problem is to select the suitable finite dimensional space V_h. In FEM these global spaces are obtained by patching together many local finite dimensional spaces (e.g. polynomial spaces) defined on the elements of a specific mesh of the domain Ω, called triangulation. As emphasized in [23], FEM is a Galerkin method characterized by the way of the construction of the finite dimensional spaces in the discrete variational formulation. The construction of finite element spaces consists of tree steps:

1. The triangulation of the domain Ω.

2. The choice of space V_h as a space of piecewise polynomials.

3. The choice of a basis $(\varphi_i)_{1 \le i \le dim(V_h)}$ of V_h with small support. If $V_h \subset V$ we obtain a conforming FEM, otherwise the FEM is a non-conforming one.

The basic finite element convergence results will be presented in Section 1.3.

1.2 Basic definitions

All the results presented in this thesis are based on the Definition 1.2.1 of a triangulation of the domain Ω and the definition of finite elements as given in the monographs [23], [31]. We use simplicial triangulation in $\mathbb{R}^d$. In the case $d = 2$ the definition of a triangulation of the domain $\Omega \subset \mathbb{R}^2$ using triangular finite elements is:

Definition 1.2.1. *A triangulation* $\mathcal{T}_h$ *is a partition of the open domain* Ω *into a finite number of triangles* τ_i *satisfying the following conditions:*

1. $\overline{\Omega} = \bigcup_{\tau_i \in \mathcal{T}_h} \tau_i.$

2. *The intersection $\overline{\tau}_1 \cap \overline{\tau}_2$ of two non-identical triangles is either empty, or a common vertex, or a common edge.*

The real parameter h characterizes the triangulation in the sense that $h := \max_{\tau \in \mathcal{T}_h} h_\tau$, where h_τ is the diameter of the triangle τ.
One of the triangulation quality measure is the aspect ratio σ_τ of an element τ of the triangulation (see [37]). It is defined as the ratio of the diameter of its circumscribed and inscribed circles:

$$\sigma_\tau := \frac{h_\tau}{\rho_\tau} \tag{1.2.1}$$

In order to control the aspect ratio independent of τ and h was introduced the concept of *shape-regularity* of a triangulation.

Definition 1.2.2. *([34]) A triangulation $\mathcal{T}_h$ is said to be shape-regular if there exists a constant σ_0 such that*

$$\sigma_\tau \leq \sigma_0, \ \forall \tau \in \mathcal{T}_h. \tag{1.2.2}$$

Definition 1.2.3. *A finite element is a triple (K, Π_K, Σ_K), where*

1. *$K \subset \mathbb{R}^d$, $\overset{\circ}{K} \neq \emptyset$ and K has a Lipschitz continuous boundary.*

2. *Π_K is a finite dimensional space of real-valued functions defined over K.*

3. *Σ_K is a Π_k unisolvent set of linear functionals (called degrees of freedom of the finite element) on Π_k, in the sense that each $p \in \Pi_K$ is uniquely defined by the functionals from Σ_K.*

We assume that each element K is obtained by a reference finite element $\hat{K}$ by means of an invertible affine map $F_K : \hat{K} \to K$.
In Chapters 3 and 4 we consider K as a triangle, the reference triangle being the unit triangle $\hat{\tau}$ with vertices $(0,0)^T$, $(1,0)^T$, $(0,1)^T$.

1.3 Basic convergence results

Let V be a Hilbert space, V_h be a finite dimensional subspace of finite element functions. First we will consider $V_h \subset V$, which corresponds to conforming case of FEM. The simplest convergence result is represented by Céa's Lemma which is the discrete analogue of Lax-Milgram's Lemma:

Theorem 1.3.1. *(Céa's Lemma - [44], [14]) Let $V_h \subset V$ be a finite dimensional subspace, $a : V \times V \to \mathbb{R}$ a bounded, coercive, bilinear form and $L \in V'$. Then the discrete variational problem (1.1.7) has an unique solution. Moreover, if u is the exact solution of the variational problem (1.1.1) then :*

$$\|u - u_h\|_V \leq \frac{C}{\alpha} \inf_{v_h \in V_h} \|u - v_h\|_V, \tag{1.3.1}$$

where C and α are the constants from the Definition 1.1.2.

Let us consider the variational problem (1.1.1). Assume that the conditions of the Lax-Milgram Lemma are satisfied and denote by u the unique solution of this variational equation. If in the discrete problem defined in (1.1.7) we approximate the bilinear form a and the linear form L by bounded bilinear form $a_h : V_h \times V_h \to \mathbb{R}$ and bounded linear form $L_h : V_h \to \mathbb{R}$, respectively and we approximate the unique exact solution u with the solution u_h of the variational equation

$$a_h(u_h, v_h) = L_h(v_h), \text{ for all } v_h \in V_h, \tag{1.3.2}$$

then the upper bound of the global discretization error is given by Strang's first Lemma.

Theorem 1.3.2. *(First Strang Lemma - [23]) Let $a : V \times V \to \mathbb{R}$ be a bilinear form satisfying the conditions of the Lax-Milgram Lemma. If $a_h(\cdot,\cdot) : V_h \times V_h \to \mathbb{R}$ is a bilinear form satisfying the Lax-Milgram Lemma and u and u_h are the unique solutions of the variational equations (1.1.1) and (1.3.2) respectively then there exists a constant $C > 0$ such that*

$$\begin{aligned}\|u-u_h\|_V \le C\Bigg(\inf_{v_h\in V_h}\Bigg(\|u-v_h\|_V + \sup_{w_h\in V_h}\frac{|a(v_h,w_h)-a_h(v_h,w_h)|}{\|w_h\|_V}\Bigg) + \\ + \sup_{w_h\in V_h}\frac{|L(w_h)-L_h(w_h)|}{\|w_h\|_V}\Bigg)\end{aligned} \tag{1.3.3}$$

Remark 1.3.3. The upper bound of the global discretization error consists of two parts:

1. The approximation error:

$$\inf_{v_h\in V_h}\|u-v_h\|_V \tag{1.3.4}$$

The approximation error depends on the selection of the space V_h and the regularity of the exact solution.

2. The consistency errors:

$$\inf_{v_h\in V_h}\sup_{w_h\in V_h}\frac{|a(v_h,w_h)-a_h(v_h,w_h)|}{\|w_h\|_V} \tag{1.3.5}$$

$$\sup_{w_h\in V_h}\frac{|L(w_h)-L_h(w_h)|}{\|w_h\|_V} \tag{1.3.6}$$

The consistency errors stem from the discretization of the equation, e.g. by numerical quadrature or non-conforming methods. It measures the consistency between the continuous and discrete equations.
The estimates of the consistency errors depends on the shape-regularity of the triangulation.

In the case of a non-conforming finite element space V_h with $V_h \not\subset V$ the upper bound of global discretization error is given by Strang's second Lemma.
For the non-conforming case we suppose that the linear functional L_h is well defined on $V_h + V$ and we denote by $\|\cdot\|_h$ a mesh-dependent norm of $V_h + V$. The approximate bilinear form of the bilinear form $a(\cdot,\cdot)$, denoted by $a_h(\cdot,\cdot)$, with $a_h(\cdot,\cdot) : (V_h + V) \times (V_h + V) \to \mathbb{R}$, is uniformly V_h-elliptic and uniformly bounded in the sense of the following definition.

Definition 1.3.4. *The bilinear form $a_h(\cdot,\cdot)$, with $a_h(\cdot,\cdot) : (V_h + V) \times (V_h + V) \to \mathbb{R}$ is:*

- *Uniformly V_h-elliptic if there is a constant $\alpha > 0$, independent of h such that*

$$a_h(v_h, v_h) \ge \alpha\|v_h\|^2, \ v_h \in V_h.$$

- *Uniformly bounded on $V_h + V$ if there is a constant M, independent of h such that*

$$|a_h(u,v)| \leq M\|u\|_h\|v\|_h, \ u,v \in V_h + V.$$

Theorem 1.3.5. *(Second Strang Lemma - [23]) Let $a : V \times V \to \mathbb{R}$ be a bilinear form satisfying the conditions of the Lax-Milgram Lemma. Let $a_h(\cdot,\cdot)$ be a bounded and uniformly V_h-elliptic approximate bilinear form defined on $(V_h+V)\times(V_h+V)$ and denote by u and u_h the unique solutions of the variational equations (1.1.1) and (1.3.2) respectively. Then there exists a constant $C > 0$ such that*

$$\|u-u_h\|_h \leq C\left(\inf_{v_h\in V_h}\|u-v_h\|_h + \sup_{v_h\in V_h}\frac{|a_h(u,v_h)-L_h(v_h)|}{\|v_h\|_h}\right). \tag{1.3.7}$$

Remark 1.3.6. The first term in (1.3.7) represents the approximation error and the second one represents the consistency error. The last term vanishes when $V_h \subset V$.

Both Strang lemmas are generalizations of Céa's Lemma.
The non-conformity is a "variational crime" in the sense of Strang [56], meaning that non-conforming elements do not belong to the class of the classical Ritz method. The non-conformity is a violation related to the continuity between adjacent elements, as explained and studied in [57]. It was proved later that non-conforming elements supply more simple constructions of stable pairs of finite element spaces in Stokes problems ([30, 40]). Crouzeix and Raviart developed a linear and a cubic non-conforming element on triangles and a linear non-conforming element on tetrahedrons.

1.4 Examples

In Chapter 3 and 4 we illustrate the intrinsic approach proposed in this thesis for Poisson's equation and linearized elasticity problem. The aim of this section is to recall these two particular problems*.

1.4.1 Poisson's equation

Poisson's equation appears as a model problem for a large number of problems from electromagnetism, hydrodynamics or mechanical engineering like: heat conduction, transmission problems, fluid flow, linearized elasticity problems, electric and magnetic fields, gravitational potential, water waves. It is also a building block to solve more complicated systems of PDEs (Navier Stokes equations). Image reconstruction from gradients is also based on Poisson's equation ([55]).
Let Ω be a bounded, open domain in $\mathbb{R}^d$ with Lipschitz continuous boundary. We will consider Poisson's problem with homogeneous Dirichlet boundary condition:
Find $u \in H_0^1(\Omega)$ such that

$$\begin{aligned} -\Delta u &= f, \ \textit{in } \Omega, \\ u &= 0, \ \textit{on } \Gamma. \end{aligned}$$

*All the notations from this section are explained in Chapter 2.

The corresponding variational formulation is:
Find $u \in H_0^1(\Omega)$ *such that*

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega f \cdot v, \ \textit{for all } v \in H_0^1(\Omega).$$

According to the Lax-Milgram Lemma the variational equation has an unique solution $u \in H_0^1(\Omega)$.
The equivalent minimization problem is:
Find $u \in H_0^1(\Omega)$ *such that*

$$J(u) = \inf_{v \in H_0^1} J(v), \ \textit{with } J(v) = \frac{1}{2}\int_\Omega \nabla u \cdot \nabla v \, dx - \int_\Omega f \cdot v \, dx. \tag{1.4.1}$$

1.4.2 Linearized elasticity equations

Let Ω be an open, bounded, connected subset of $\mathbb{R}^3$, with a Lipschitz continuous boundary Γ consisting of two disjoint pieces, $\Gamma_D \cap \Gamma_T = \emptyset$. To fix ideas we consider in the following a homogeneous, isotropic, linear elastic body. The linearized elasticity problem consists in finding the linearized displacement tensor field $\mathbf{u}$ inside an elastic body, which was subject to a deformation, governed by the equilibrium equations ([10, 35]):

$$-\mathbf{div}(\sigma(\mathbf{u})) = \mathbf{f}, \text{ in } \Omega, \tag{1.4.2a}$$
$$\mathbf{u} = \mathbf{g}, \text{ on } \Gamma_D, \tag{1.4.2b}$$
$$\sigma(\mathbf{u}) \cdot \mathbf{n} = h, \text{ on } \Gamma_T. \tag{1.4.2c}$$

with

$$\sigma(\mathbf{u}) = 2\mu\mathbf{e}(\mathbf{u}) + \lambda tr(\mathbf{e}(\mathbf{u}))\mathbf{I}, \tag{1.4.3a}$$
$$\mathbf{e}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T). \tag{1.4.3b}$$

The gradient operator ∇ is defined in Section 2.2.1. The strain tensor is denoted by $\mathbf{e}$ and σ represents the stress tensor.
We use boldface letters to denote vector fields and spaces of vector fields. The pure displacement linearized elasticity problem is obtained for $\Gamma_D = \emptyset$ and pure traction linearized elasticity problem is obtain for $\Gamma_T = \emptyset$. The reference configuration of the elastic body in the absence of applied forces is $\overline{\Omega}$ and λ, $\mu > 0$ are its Lamé moduli*. For most materials $\lambda > 0$. If the material of the elastic body is homogeneous λ and μ are constants. The body is subject of applied forces of density $\mathbf{f} \in \mathbf{L}^{6/5}(\Omega)$ in its interior and $\mathbf{g} \in \mathbf{L}^{4/3}(\Gamma_D)$ on its boundary. The assumed regularity on $\mathbf{f}$ and $\mathbf{g}$ is necessary to ensure the continuity of the linear form ([24])

$$L(\mathbf{v}) := \int_\Omega \mathbf{f} \cdot \mathbf{v} \ dx + \int_\Gamma \mathbf{g} \cdot \mathbf{v} \ d\Gamma, \ \forall \ \mathbf{v} \in \mathbf{H}^1(\Omega). \tag{1.4.4}$$

The elastic body is characterized by its elasticity tensor $\mathbf{A} = (A_{ijkl}) \in \mathbf{L}^\infty(\Omega)$, which is symmetric, that is $A_{ijkl} = A_{jikl} = A_{klij}$ and is uniformly positive-definite in Ω, meaning that there exists $\alpha > 0$ such that

$$\mathbf{A}(x)\mathbf{t} : \mathbf{t} \geq \alpha\mathbf{t} : \mathbf{t}$$

*More assumptions on Lamé moduli are given in Chapter 4

for almost all $x \in \Omega$ and $\mathbf{t} \in \mathbb{S}^3$ and $(\mathbf{A}(x)\mathbf{t})_{ij} := A_{ijkl}(x)t_{kl}$ (c.f. [24, 26, 25]). We denote by $\mathbf{a} : \mathbf{b}$ the Frobenius inner product of two matrices $\mathbf{a}$ and $\mathbf{b}$.
For any matrix $\mathbf{e} = (e)_{ij} \in \mathbb{S}^d$ we define the matrix $\mathbf{Ae} \in \mathbb{S}^d$ as in [26] by

$$\mathbf{Ae} = 2\mu\mathbf{e} + \lambda tr(\mathbf{e})\mathbf{I}.$$

Given a vector field $\mathbf{v} = (v_i) \in \mathbf{H}^1(\Omega)$, we denote by $\nabla_s\mathbf{v} := \frac{1}{2}(\nabla\mathbf{v} + (\nabla\mathbf{v})^T)$ the symmetric gradient of $\mathbf{v}$.
The elements of the elasticity tensor are given by:

$$A_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$

that is

$$\mathbf{Ae} = 2\mu\mathbf{e}(\mathbf{u}) + \lambda tr(\mathbf{e}(\mathbf{u}))\mathbf{I}. \tag{1.4.5}$$

The pure traction problem of linearized elasticity consists in finding the linearized displacement field $\mathbf{u}$ from the equations

$$\begin{aligned} -\mathbf{div}(\sigma) &= \mathbf{f}, \text{ in } \Omega, && (1.4.6a)\\ \mathbf{u} &= \mathbf{g}, \text{ on } \Gamma_D, && (1.4.6b)\\ \sigma &= \mathbf{Ae}, && (1.4.6c)\\ \mathbf{e} &= \nabla_s\mathbf{u}. && (1.4.6d) \end{aligned}$$

The variational formulation of the pure traction problem consists in finding $\dot{\mathbf{u}} \in \dot{\mathbf{H}}^1(\Omega) := \mathbf{H}^1/\mathbf{ker}\nabla_s$, which solves the variational equation

$$\int_\Omega \mathbf{A}\nabla_s\dot{\mathbf{u}} : \nabla_s\dot{\mathbf{v}} = L(\dot{\mathbf{v}}), \quad \forall\dot{\mathbf{v}} \in \dot{\mathbf{H}}^1(\Omega). \tag{1.4.7}$$

The minimization problem which is equivalent to (1.4.7) consists in finding $\dot{\mathbf{u}} \in \dot{\mathbf{H}}^1(\Omega)$ such that:

$$J(\dot{\mathbf{u}}) = \inf_{\dot{\mathbf{v}} \in \dot{\mathbf{H}}^1(\Omega)} J(\dot{\mathbf{v}}), \text{ where } J(\dot{\mathbf{v}}) = \frac{1}{2}\int_\Omega \mathbf{A}\nabla_s\dot{\mathbf{v}} : \nabla_s\dot{\mathbf{v}} - L(\dot{\mathbf{v}}), \tag{1.4.8}$$

$L(\mathbf{v})$ is defined in (1.4.4). The variational problem (1.4.7) and the equivalent form (1.4.8) have a solution if and only if the compatibility condition

$$L(\mathbf{v}) = 0 \tag{1.4.9}$$

is satisfied for all $\mathbf{v} \in \mathbf{R}(\Omega)$, where $\mathbf{R}(\Omega)$ is the space of infinitesimal rigid displacement fields of Ω which represents $\mathbf{ker}\nabla_s$:

$$\mathbf{R}(\Omega) := \{\mathbf{r} \in \mathbf{H}^1(\Omega); \nabla_s(\mathbf{r}) = \mathbf{0} \text{ in } \Omega\} \tag{1.4.10}$$

If the compatibility condition (1.4.9) is satisfied the solution is unique ([33]).

The pure displacement problem of linearized elasticity
The variational formulation of the pure displacement problem of linearized elasticity consists in finding the displacement vector field $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$, solution of the variational equation

$$\int_\Omega \mathbf{A}\nabla_s\mathbf{u} : \nabla_s\mathbf{v} = L(\mathbf{v}), \ \forall\mathbf{v} \in \mathbf{H}_0^1(\Omega). \tag{1.4.11}$$

The minimization problem which is equivalent to (1.4.11) consists in finding the displacement vector field $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$ such that:

$$J(\mathbf{u}) = \inf_{v \in \mathbf{H}_0^1(\Omega)} J(\mathbf{v}), \text{ where } J(\mathbf{v}) = \frac{1}{2} \int_\Omega \mathbf{A} \nabla_s \mathbf{v} : \nabla_s \mathbf{v} dx - L(\mathbf{v}), \tag{1.4.12}$$

with $L(\mathbf{v})$ given in (1.4.4). In the case of the pure displacement problem there is no compatibility condition imposed on the linear functional L, because $\mathbf{ker} \nabla_{\mathbf{s}} = \{0\}$ in $\mathbf{H}_0^1(\Omega)$.

2

Intrinsic methods in FEM

2.1 General framework

The main idea of the intrinsic approach is to compute directly physical quantities which otherwise are obtained by numerical differentiation from the primary unknown of the problem. More precisely, depending on the problem, instead of the potential u or the displacement vector $\mathbf{v}$ the primary unknown is considered to be the gradient field $\mathbf{e} = \nabla(u)$, the symmetric gradient tensor $\mathbf{e} = \nabla_s(\mathbf{v})$, the Green-Saint Venant tensor $\mathbf{E}(\mathbf{v}) = \left(\nabla\mathbf{v}^T + \nabla\mathbf{v} + \nabla\mathbf{v}^T\nabla\mathbf{v}\right)/2$ or the Cauchy-Green tensor $\mathbf{I} + 2\mathbf{E}(\mathbf{v})$. The method has numerous applications where one is interested to find physical quantities like the flux, the electrostatic and the magnetic fields, the velocity field or the strain tensor field. The direct computation is preferable in order to avoid the loss of accuracy by numerical differentiation. The change of the primary unknown raises a series of questions related to the possibility of adapting the classical known results from FEM theory to the intrinsic theory, questions that we highlight and explain in this chapter.

The intrinsic method is different from the mixed methods. In the mixed methods fields of different types appear together in the weak formulation of the variational problem ([9, 11, 13, 42, 47, 46, 32]).

In the development of an intrinsic FEM approach we start from the weak variational formulation (1.1.1) of the classical problem. We refer here to the monograph of P.G. Ciarlet [23] and Schwab [53]. For the classical FEM theory of shells and plates we refer to the [39, 15]. Then the existence and uniqueness of the solution in the space V from (1.1.2) is proved, typically by using the Lax-Milgram Lemma. The reformulation of the problem in an intrinsic way requires the characterization of the finite element space as a subspace of the energy space and the reformulation of the energy minimization in terms of the energy space. We will derive appropriate compatibility conditions for this purpose and define the isomorphism between the involved spaces.

In the following sections we will detail the general framework in order to obtain intrinsic discretizations for different problems. In Section 2.2 we introduce usual notations and main operators for the development of our intrinsic approach presented in Chapters 3 and 4. Sections 2.3 and 2.4 are devoted to the characterization of vector and matrix fields. The discretization process for FEM intrinsic approaches is presented in Section 2.5. Miscellaneous remarks concerning the convergence and the analogy between vector and matrix cases can be found in Section 2.6.

2.2 Notations and formulas

2.2.1 General notations

Throughout the thesis we denote by Ω a domain in $\mathbb{R}^d$, $d \in \{2,3\}$; by $H^1(\Omega)$ the usual Sobolev space which contains $L^2(\Omega)$ functions with weak first derivatives in $L^2(\Omega)$, and by $H_0^1(\Omega) \subset H^1(\Omega)$ the subspace of those functions in $H^1(\Omega)$ with zero traces at the boundary Γ of Ω. The dual space of $H_0^m(\Omega)$ is denoted by $H^{-m}(\Omega)$. For $k \in \mathbb{N}_0$ let $\mathcal{P}^k$ be the space of polynomials of maximal total degree k.
The vector fields and spaces of vector fields are indicated by boldface letters (e.g. $\mathbf{H}^1(\Omega)$) and the matrix field and spaces of matrix-valued functions by capital roman letters (such as $\mathbb{L}^2(\Omega)$, $\mathbb{P}^k(\Omega)$, $\mathbb{E}(\Omega)$).
For the space of all symmetric matrices of order 2 we use the notation $\mathbb{S}$ and the spaces of symmetric vector and tensor fields are denoted using a subscript s (e.g. $\mathbb{P}_s^p(\Omega) = \mathbb{P}^p(\Omega;\mathbb{S})$ is the space of symmetric matrices of order 2 whose elements are polynomials in two variables of total degree $\leq p$; $\mathbb{L}_s^2(\Omega) = L^2(\Omega;\mathbb{S})$, $\mathbf{H}_s^1(\Omega) = \mathbf{H}^1(\Omega,\mathbb{S})$).
The quotient space of a vector space V modulo the subspace $R \subseteq V$ is denoted by V/R.
A point $\mathbf{x} \in \mathbb{R}^2$, is given by its coordinates x_i, $i \in \{1,2\}$ and we denote $\partial_i := \dfrac{\partial}{\partial x_i}$ and $\partial_{ij} := \dfrac{\partial^2}{\partial x_i \partial x_j}$.
For a scalar mapping $u \in H^1(\Omega)$, the gradient of u is defined as the column vector

$$\nabla u := (\partial_j u)_{j=1}^d \tag{2.2.1}$$

Given any vector field $\mathbf{v} = (v_i)_{i=1,\ldots,d} \in \mathbf{H}^1(\Omega)$, the flux or gradient of $\mathbf{v}$ is defined as the matrix

$$\nabla \mathbf{v} := \begin{bmatrix} \partial_1 v_1 & \ldots & \partial_d v_1 \\ \vdots & & \vdots \\ \partial_1 v_d & \ldots & \partial_d v_d \end{bmatrix} \tag{2.2.2}$$

and

$$\nabla_s \mathbf{v} := \frac{1}{2}(\nabla \mathbf{v} + (\nabla \mathbf{v})^T) \in \mathbb{L}_s^2(\Omega) \tag{2.2.3}$$

represents the symmetric gradient of $\mathbf{v}$.
For a continuously differentiable vector field $\mathbf{v}$ the divergence, $\mathrm{div}(\mathbf{v})$ is defined as the scalar field:

$$\mathrm{div}(\mathbf{v}) = \nabla \cdot \mathbf{v}. \tag{2.2.4}$$

For a continuously differentiable matrix field $\mathbf{e} = (e_{ij})$ the vector divergence, $\mathbf{div}(\mathbf{e})$ is defined as the vector field:

$$\mathbf{div}(\mathbf{e}) = \begin{bmatrix} \partial_1 e_{11} + \ldots + \partial_d e_{1d} \\ \ldots \\ \partial_1 e_{d1} + \ldots + \partial_d e_{dd} \end{bmatrix}. \tag{2.2.5}$$

Let $\mathbf{a} : \mathbf{b}$ denote the Frobenius inner product of two matrices $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{u} \cdot \mathbf{v}$ the Euclidian scalar product of two vectors $\mathbf{u}$ and $\mathbf{v}$ and the product between a matrix $\mathbf{m}$ and a vector $\mathbf{v}$ is simply denoted by $\mathbf{mv}$.
We introduce some notation for the exterior calculus in two dimensions.
Exterior product: For vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$ we set

$$\mathbf{a} \times \mathbf{b} := a_1 b_2 - a_2 b_1 \tag{2.2.6}$$

Variants of the curl operator: To write partial integration formulas in a compact way we introduce some variants of the curl operator.
The scalar curl operator: Given a differentiable vector field $\mathbf{w} : \Omega \to \mathbb{R}^2$ we define its scalar curl by

$$\operatorname{curl}(\mathbf{w}) := \nabla \times \mathbf{w} = \partial_1 w_2 - \partial_2 w_1 \tag{2.2.7}$$

The vector curl operator: The vector curl operator **curl** is defined for v being a scalar, differentiable function by

$$\mathbf{curl}\,(v) := \begin{pmatrix} \partial_2 v \\ -\partial_1 v \end{pmatrix}. \tag{2.2.8}$$

For a 2×2 matrix $\mathbf{m} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$ and a vector $\mathbf{a} \in \mathbb{R}^2$ we set:

$$\mathbf{a} \times \mathbf{m} : = \begin{pmatrix} a_1 m_{12} - a_2 m_{11} \\ a_1 m_{22} - a_2 m_{21} \end{pmatrix},$$

so that for a differentiable mapping $\mathbf{e} : \Omega \to \mathbb{R}^{2\times 2}$ the vector **curl** operator is given by

$$\mathbf{curl}\,(\mathbf{e}) := \nabla \times \mathbf{e} = \begin{pmatrix} \partial_1 e_{12} - \partial_2 e_{11} \\ \partial_1 e_{22} - \partial_2 e_{21} \end{pmatrix}. \tag{2.2.9}$$

The matrix curl operator: The matrix curl operator **CURL** is given for differentiable vector fields $\mathbf{w} : \Omega \to \mathbb{R}^2$ by

$$\mathbf{CURL}\,(\mathbf{w}) := \begin{bmatrix} \partial_2 w_1 & \partial_2 w_2 \\ -\partial_1 w_1 & -\partial_1 w_2 \end{bmatrix}. \tag{2.2.10}$$

By repeated application of different curl-operators we obtain the following:
For a twice differentiable mapping $\mathbf{e} : \Omega \to \mathbb{S}$ we have:

$$\operatorname{curl} \mathbf{curl}(\mathbf{e}) = \partial_{11} e_{22} - 2\partial_{12} e_{12} + \partial_{22} e_{11}.$$

For a scalar differentiable function v we obtain:

$$\mathbf{CURL}(\mathbf{curl}(v)) = \begin{pmatrix} \partial_{22} v & -\partial_{12} v \\ -\partial_{12} v & \partial_{11} v \end{pmatrix}.$$

Let $\mathcal{T}$ be a triangulation of the domain Ω in the sense of Ciarlet (see Definition 1.2.1 and Section 2.2.2). For later purposes we define the triangle-wise **curl** and triangle-wise curl **curl** operators by:

$$\left.\begin{aligned} \mathbf{curl}_{\mathcal{T}}(\mathbf{e})(\mathbf{x}) &:= \mathbf{curl}(\mathbf{e})(\mathbf{x}) \\ \operatorname{curl}_{\mathcal{T}} \mathbf{curl}_{\mathcal{T}}(\mathbf{e})(\mathbf{x}) &:= \operatorname{curl} \mathbf{curl}(\mathbf{e})(\mathbf{x}) \end{aligned}\right\} \qquad \forall \mathbf{x} \in \Omega \backslash \left(\bigcup_{E \in \mathcal{E}} E \right),$$

where $\mathcal{E}$ is the set of all interior edges of the triangulation (see Section 2.2.2).

2.2.2 Notations related to triangulation

The finite element method, both in classical or intrinsic approach is based on triangulations, or meshes, $\mathcal{T}$ of Ω. In this thesis we consider that the triangulations are regular in the sense of [23] (see also Chapter 1). As a convention we assume that a triangle is a closed set and the edges are also closed sets. The interior of a triangle τ is denoted by $\overset{\circ}{\tau}$ and we write $\overset{\circ}{E}$ for the relative interior of an edge E. The set of all interior edges is denoted by $\mathcal{E}$ and the set of edges lying on $\partial\Omega$ is $\mathcal{E}_{\partial\Omega}$. The set of interior vertices is $\mathcal{V}$ and the set of vertices lying on $\partial\Omega$ is $\mathcal{V}_{\partial\Omega}$.
For any $E \in \mathcal{E}$, we set

$$\mathcal{T}_E := \{\tau \in \mathcal{T} : E \subset \partial\tau\} \qquad \omega_E := \bigcup_{\tau\in\mathcal{T}_E} \tau. \tag{2.2.11}$$

For any $V \in \mathcal{V}$, we set

$$\mathcal{E}_V := \{E \in \mathcal{E} : V \in \partial E\}, \ \mathcal{T}_V := \{\tau \in \mathcal{T} : V \in \tau\}, \ \omega_V := \bigcup_{\tau\in\mathcal{T}_V} \tau, \tag{2.2.12}$$

and for any $\tau \in \mathcal{T}$ we denote

$$\omega_\tau = \tau. \tag{2.2.13}$$

Figure 2.1 illustrates different types of triangle patches.

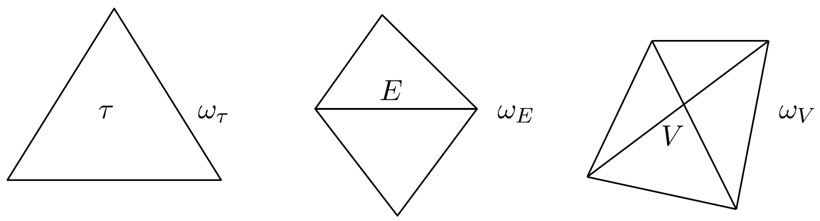

Figure 2.1: *The domains ω_τ, ω_E and ω_V*

For setting up the standard basis functions for hp-finite element spaces (cf. [53]) we introduce the set of nodal points of polynomial order p:

$$\widehat{\mathcal{N}}^p := \left\{ \frac{(i,j)^{\mathrm{T}}}{p} : (i,j) \in \mathbb{N}_0^2 \text{ with } i+j \leq p \right\} \tag{2.2.14}$$

denotes the equispaced unisolvent set of nodal points on the unit triangle $\hat{\tau}$ defined as the triangle with vertices $(0,0)^{\mathrm{T}}$, $(1,0)^{\mathrm{T}}$, $(0,1)^{\mathrm{T}}$. For a triangle $\tau \in \mathcal{T}$ with vertices $\mathbf{A}^\tau$, $\mathbf{B}^\tau$, $\mathbf{C}^\tau$, (cf. Figure 2.2) let $\chi_\tau : \hat{\tau} \to \tau$ denote the affine mapping

$$\chi_\tau(\hat{\mathbf{x}}) := \mathbf{A}^\tau + (\mathbf{B}^\tau - \mathbf{A}^\tau)\,\hat{x}_1 + (\mathbf{C}^\tau - \mathbf{A}^\tau)\,\hat{x}_2. \tag{2.2.15}$$

Then, the set of interior nodal points is given by

$$\mathcal{N}^p := \left\{ \chi_\tau\left(\hat{N}\right) \mid \hat{N} \in \widehat{\mathcal{N}}^p, \tau \in \mathcal{T} \right\} \backslash \partial\Omega. \tag{2.2.16}$$

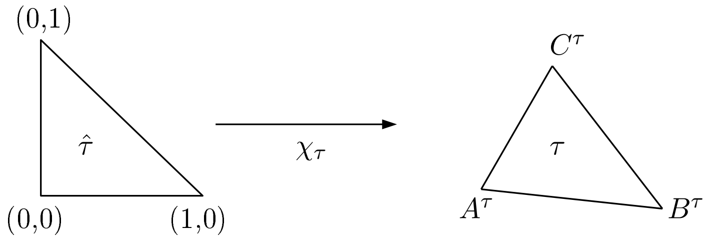

Figure 2.2: *The correspondence between the reference triangle $\hat{\tau}$ and the triangle τ*

The Lagrange basis $\{b^{\mathcal{T}}_{p,N},\ N \in \mathcal{N}^p\}$ for the classical *hp*-finite element spaces can be indexed by the nodal points $N \in \mathcal{N}^p$ and is characterized by

$$\forall N' \in \mathcal{N}^p \quad b^{\mathcal{T}}_{p,N}\left(N'\right) = \begin{cases} 1, & N = N', \\ 0, & N \neq N'. \end{cases} \tag{2.2.17}$$

2.3 Characterization of continuous energy spaces

First we have to characterize the continuous energy spaces for the direct computation of the fluxes instead of the potential. In [18] we consider the problem of finding directly the electrostatic field $\mathbf{e}$, as a solution of the boundary value problem $-\operatorname{div}(\varepsilon\mathbf{e}) = \rho$, in Ω and $\mathbf{e}|_{\partial\Omega} = 0$, instead of computing first the scalar potential u, with $\mathbf{e} = \nabla(u)$. The electrostatic potential u is chosen to satisfy the condition $u|_{\partial\Omega} = 0$. The problem is complemented with a perfect conductor boundary condition and the domain Ω is supposed to be a bounded Lipschitz domain with connected boundary Γ. This corresponds theoretically in finding an intrinsic approach for the discretization of Poisson's equation. To this end we need to characterize the vector fields $\mathbf{e} \in \mathbf{L}^2(\Omega)$, which can be written as $\mathbf{e} = \nabla(u)$ for some scalar field $u \in H^1(\Omega)$. The characterization will be unique up to an element from $\ker(\nabla)$, that is up to a constant. The connectivity of the domain Ω is required.
It is proved (see e.g. [1]) that the curl-free condition, $\operatorname{curl}\mathbf{e} = 0$, represents one of the characterization conditions for the energy space. The proof is based on the following generalization of Poincaré's theorem:

Theorem 2.3.1. *(Poincaré's theorem in $\mathbf{H}^{-1}(\Omega)$ - [38, 3]) Let $\Omega \subset \mathbb{R}^3$ be a simply-connected domain. If $\mathbf{h} \in \mathbf{H}^{-1}(\Omega)$ satisfies the condition $\operatorname{curl}\mathbf{h} = 0$ in $\mathbf{H}^{-2}(\Omega)$ then there exists $p \in L^2(\Omega)$ such that $\mathbf{h} = \nabla p$ in $H^{-1}(\Omega)$.*

In Chapter 3 we use this compatibility condition for the energy space in the form given in the next theorem.

Theorem 2.3.2. *Let $\Omega \subset \mathbb{R}^2$ be an open, bounded domain with connected Lipschitz boundary Γ. If $\mathbf{e} \in \mathbf{L}^2(\Omega)$ satisfies the condition $\operatorname{curl}\mathbf{e} = 0$ in $\left(H^1(\Omega)\right)'$ then there exists $u \in H^1(\Omega)$ unique up to a constant function, such that $\mathbf{e} = \nabla(u)$.*

Based on the results presented before, we define (see Chapter 3) the space

$$\mathbf{E}(\Omega) = \left\{\mathbf{e} \in \mathbf{L}^2(\Omega) \mid \operatorname{curl}\mathbf{e} = 0 \text{ in } H^{-1}(\Omega) \text{ and } \mathbf{e} \times \mathbf{n} = 0 \text{ in } H^{-1/2}(\Gamma)\right\}.$$

It is proved in [43] that the linear operator $\nabla : H_0^1(\Omega) \to \mathbf{E}(\Omega)$ is an isomorphism and thus its inverse operator $\Lambda : \mathbf{E}(\Omega) \to H_0^1$ is continuous. Thanks to the isomorphism Λ we can formulate the intrinsic variational form of the problem (3.2.1), whose formulation is given in (3.2.6). The equivalent minimization problem is formulated in (3.2.7). For convenience we rewrite next these two equivalent formulations:
*The intrinsic variational problem:** *Find* $\mathbf{e} \in \mathbf{E}(\Omega)$ *such that*

$$\int_\Omega \varepsilon \mathbf{e} \cdot \tilde{\mathbf{e}} = \int_\Omega \rho \Lambda \tilde{\mathbf{e}}, \qquad \forall \tilde{\mathbf{e}} \in \mathbf{E}(\Omega).$$

The intrinsic minimization problem: Find $\mathbf{e} \in \mathbf{E}(\Omega)$ *such that*

$$j(\mathbf{e}) = \inf_{\tilde{e} \in \mathbf{E}} j(\tilde{\mathbf{e}}), \text{ with } j(\tilde{\mathbf{e}}) := \frac{1}{2}\int_\Omega \varepsilon \tilde{\mathbf{e}} \cdot \tilde{\mathbf{e}} - \int_\Omega \rho \Lambda \tilde{\mathbf{e}}.$$

Taking into account Remark 1.1.1, the two problems formulated before are equivalent.
Since the conditions of the Lax-Milgram Lemma are satisfied, these equivalent problems have an unique solution in $\mathbf{E}(\Omega)$.

2.4 Characterization of matrix fields

Characterization conditions are necessary to describe the energy space for elasticity in an intrinsic way. Intrinsic FEM approaches in linearized elasticity consists in the direct approximation of the linearized strain tensor instead of the displacement field. To this end we need to characterize the tensor fields $\mathbf{e} \in \mathbf{L}^2(\Omega)$, which can be written as $\mathbf{e} = \nabla_s(\mathbf{u})$ for some vector field $\mathbf{u} \in \mathbf{H}^1(\Omega)$. The characterization will be unique up to an element from $\ker(\nabla_s)$, that is up to a infinitesimal rigid displacement field $\mathbf{r} \in \mathbf{R}(\Omega)$ (see also Chapters 1 and 4).
There are two types of characterization conditions used in intrinsic FEM in linearized elasticity: Saint Venant's and Donati's compatibility conditions, each of which leads to different finite element spaces.

2.4.1 Saint Venant's characterization conditions

Saint Venant's characterization conditions are based on Saint Venant's theorems. The theorem of Saint Venant was stated in 1864, but a rigorous proof was formulated only in 1886 by E. Beltrami.

Theorem 2.4.1. *(Saint Venant's theorem - [4]) Let* Ω *be an open, simply-connected subset of* $\mathbb{R}^3$. *Let* $\mathbf{e} = (e_{ij})$ *with* $i, j \in \{1, 2, 3\}$. *If the functions* e_{ij} *are in the space* $C^2(\Omega)$ *and satisfy:*

$$\mathfrak{R}_{ijkl}(\mathbf{e}) := \partial_{lj} e_{ik} + \partial_{ki} e_{jl} - \partial_{li} e_{jk} - \partial_{kj} e_{il} = 0 \text{ in } \Omega, \ \forall i, \ j, \ k, \ l \in \{1, 2, 3\} \tag{2.4.1}$$

then there exists a vector field $\mathbf{u} \in \mathbf{C}^3(\Omega)$ *such that* $\mathbf{e} = \nabla_s \mathbf{u}$ *in* Ω.

The proof of Saint Venant's theorem is based on the classical Poincaré's theorem.

Theorem 2.4.2. *(Classical theorem of Poincaré - [54, 24]) Let* $\Omega \subset \mathbb{R}^n$ *be a simply-connected, open subset of* $\mathbb{R}^n$. *If* $h_k \in C^1(\Omega)$ *are functions that satisfy the condition* $\partial_l h_k = \partial_k h_l$, $k, l \in \{1, \ldots, n\}$ *then there exists* $p \in C^2(\Omega)$ *such that* $h_k = \partial_k p$.

*Assumption on ε and ρ are given in Section 3.2.

Equation (2.4.1) defines the so-called Saint Venant's characterization conditions.

Remark 2.4.3. It is proved that (c.f. [4]) the Saint Venant conditions (2.4.1) reduce to

$$\operatorname{curl}\mathbf{curl}(\mathbf{e}) = 0 \text{ in } \mathbf{H}^{-2}(\Omega). \tag{2.4.2}$$

In the case of a domain $\Omega \subset \mathbb{R}^2$ and $\mathbf{e} \in \mathbb{L}^2_s(\Omega)$ the conditions (2.4.2) reduce to only one equation ([4, 25]):

$$\partial_{11}e_{22} - 2\partial_{12}e_{12} + \partial_{22}e_{11} = 0 \quad \text{in } \mathbf{H}^{-2}(\Omega).$$

Saint Venant's theorem was extended by P.G. Ciarlet and P. Ciarlet, Jr. in 2005, in [24] in the sense of distributions:

Theorem 2.4.4. *Let Ω be an open, simply-connected subset of $\mathbb{R}^3$. If $\mathbf{e} \in \mathbb{L}^2_s(\Omega)$ satisfies the conditions (2.4.1) in $H^{-2}(\Omega)$ then there exists a vector field $\mathbf{v} \in \mathbf{H}^1(\Omega)$ such that $\mathbf{e} = \nabla_s\mathbf{v}$ in Ω. The other solutions $\tilde{\mathbf{v}}$ of equation $\mathbf{e} = \nabla_s\tilde{\mathbf{v}}$ differ by an infinitesimal rigid displacement of the domain Ω: $\tilde{\mathbf{v}} = \mathbf{v} + \mathbf{r},\ \mathbf{r} \in \mathbf{R}(\Omega)$.*

The proof is based on the generalization of Poincaré's theorem in $H^{-1}(\Omega)$ introduced and proved in the same article ([24]).

Theorem 2.4.5. *([24]) Let Ω be a bounded, connected and simply-connected open subset of $\mathbb{R}^3$ with a Lipschitz continuous boundary. If the distributions $h_k \in H^{-1}(\Omega)$ satisfy the condition $\partial_l h_k = \partial_k h_l$ in $H^{-2}(\Omega)$ then there exists an unique function $p \in L^2(\Omega)$ up to an additive constant, such that $h_k = \partial_k p$ in $H^{-1}(\Omega)$.*

Other extensions for matrix fields with components in $H^{-1}(\Omega)$ are given in [4, 1]. The extension of Saint Venant's theorem in $\mathbb{H}^{-1}(\Omega)$, that we will use in Chapter 4 for extending our intrinsic method presented in [18] to the pure traction linearized elasticity problem is given below:

Theorem 2.4.6. *(Saint Venant's theorem in $\mathbb{H}^{-1}(\Omega)$ [3, 38]) Let $\Omega \subset \mathbb{R}^3$ be a simply-connected domain. If $\mathbf{e} \in \mathbb{H}^{-1}_s(\Omega)$ satisfies the condition $\operatorname{curl}\mathbf{curl}(\mathbf{e}) = 0$ in $\mathbb{H}^{-3}(\Omega)$ then there exists $\mathbf{v} \in \mathbf{L}^2(\Omega)$ such that $\mathbf{e} = \nabla_s\mathbf{v}$. Moreover all other vector fields $\tilde{\mathbf{v}}$ satisfying $\mathbf{e} = \nabla_s\tilde{\mathbf{v}}$ differ by an infinitesimal rigid displacement of the domain Ω.*

Let us remark that Saint Venant's theorem and its extensions require the simply-connected property of the domain Ω and that the matrix field $\mathbf{e}$ is symmetric.
Due to Theorems 2.4.4 and 2.4.6 it is possible to consider the stress tensor as primary unknown of the pure traction problem of linearized elasticity instead of the displacement field.

In [24, 26], the pure traction problem of linearized elasticity is considered for a body with reference configuration $\overline{\Omega}$, Ω being an open, bounded, and connected subset of $\mathbb{R}^3$ with Lipschitz continuous boundary Γ. The body is subject to the forces $\mathbf{f} \in \mathbf{L}^{6/5}(\Omega)$ in its interior and $\mathbf{g} \in \mathbf{L}^{4/3}(\Gamma)$ on its boundary.
The intrinsic formulation of the problem is based on the space

$$\mathbb{E}_1(\Omega) := \{\mathbf{e} \in \mathbb{L}^2_s(\Omega);\ \text{such that } \operatorname{curl}\mathbf{curl}(\mathbf{e}) = 0 \text{ in } H^{-2}(\Omega)\}. \tag{2.4.3}$$

It is proved that the operator $\nabla_s : \dot{\mathbb{H}}^1(\Omega) := \mathbb{H}^1/\mathbf{R}(\Omega) \to \mathbb{E}_1(\Omega)$ is surjective, injective, and continuous and therefore its inverse operator $\mathcal{F}_1 := \nabla_s : \mathbb{E}_1(\Omega) \to \dot{\mathbb{H}}^1(\Omega)$ is an isomorphism

and allows the intrinsic reformulation of the pure traction problem (1.4.7).
Find $\mathbf{e} \in \mathbb{E}_1(\Omega)$ *such that*:

$$\int_\Omega \mathbf{A}\mathbf{e} : \tilde{\mathbf{e}} = L \circ \mathcal{F}_1(\tilde{\mathbf{e}}),\ \forall \tilde{\mathbf{e}} \in \mathbb{E}_1(\Omega). \tag{2.4.4}$$

Taking into account the Remark 1.1.1 this variational problem is equivalent with the minimization problem:
Find $\mathbf{e} \in \mathbb{E}_1(\Omega)$ *such that*:

$$J(\mathbf{e}) = \inf_{\tilde{\mathbf{e}} \in \mathbb{E}_1(\Omega)} J(\tilde{\mathbf{e}}),\ \textit{where}\ J(\tilde{\mathbf{e}}) = \frac{1}{2}\int_\Omega \mathbf{A}\tilde{\mathbf{e}} : \tilde{\mathbf{e}} - L \circ \mathcal{F}_1(\tilde{\mathbf{e}}). \tag{2.4.5}$$

It is proved in [24] that this problem has one and only one solution. Moreover this solution satisfies $\mathbf{e} = \nabla_s \mathbf{u}$, with $\mathbf{u}$ being the unique solution in $\dot{\mathbb{H}}^1(\Omega)$ of the classical problem (1.4.8). Let us remark that the minimization problem (2.4.5) can be understood as a constrained minimization problem over $\mathbb{L}^2_s(\Omega)$, the constraint being the Saint Venant compatibility conditions.

2.4.2 Donati's characterization conditions

Another characterization of matrix fields was given by Donati in 1890.

Theorem 2.4.7. *(Donati's theorem [4]) Let $\Omega \subset \mathbb{R}^3$ be an open domain. If the components e_{ij} of a symmetric matrix field $\mathbf{e} \in \mathbb{C}^2_s(\Omega)$ satisfy:*

$$\int_\Omega e_{ij}s_{ij}dx = 0, \quad \forall \mathbf{s} = (s_{ij}) \in \mathbb{D}_s(\Omega)\ \textit{such that}\ \mathbf{div}(\mathbf{s}) = 0\ \textit{in}\ \Omega \tag{2.4.6}$$

then $\operatorname{curl}\mathbf{curl}(\mathbf{e}) = 0$.
We denote by $\mathbb{D}_s(\Omega)$ the space of symmetric tensor fields whose components are infinitely differentiable and have compact support in Ω.

Remark 2.4.8. In case of a simply-connected domain, Donati's theorem combined with Saint Venant's theorem gives a new characterization of symmetric matrix fields. This characterization is given in the next corollary.

Corollary 2.4.9. *If the domain $\Omega \subset \mathbf{R}^3$ is an open simply-connected domain and the symmetric matrix field $\mathbf{e}$ satisfies the conditions (2.4.6) then there exists $\mathbf{u} \in \mathbf{C}^3(\Omega)$ such that $\mathbf{e} = \nabla_s \mathbf{u}$ in Ω.*

As in the case of Saint Venant's characterization, the characterization given in Corollary 2.4.9 is unique up to a vector field $\mathbf{r} \in \mathbf{R}(\Omega)$.
Donati's theorem has been extended by T.W.Ting [60] in 1974 for components of symmetric matrix fields in L^2 and by J.J. Moreau [45] in 1979 in the sense of distributions. Two other characterizations of Donati's type, which do not require the simply-connected condition for the domain, can be found in [4, 17]. We present next these two characterizations.

Theorem 2.4.10. *([4, 17]) Let Ω be a bounded, connected, open subset in $\mathbb{R}^3$ and $\mathbf{e} \in \mathbb{L}^2_s(\Omega)$. Let us define the space*

$$\mathbb{M} := \{\mathbf{s} \in \mathbb{L}^2_s(\Omega) \,|\, \mathbf{div}(\mathbf{s}) = \mathbf{0}\ \textit{in}\ \mathbf{H}^{-1}(\Omega))\}. \tag{2.4.7}$$

There exists a vector field $\mathbf{v} \in \mathbf{H}^1(\Omega)$ *such that* $\mathbf{e} = \nabla_s \mathbf{v}$ *if and only if*

$$\int_\Omega \mathbf{e} : \mathbf{s} = 0 \quad \textit{for all } \mathbf{s} \in \mathbb{M}. \tag{2.4.8}$$

The solution of the equation $\mathbf{e} = \nabla_s \mathbf{v}$ *is unique up to a infinitesimal rigid displacement* $\mathbf{r} \in \mathbf{R}(\Omega)$.

Theorem 2.4.11. *Let* Ω *be a bounded, connected, open subset in* $\mathbb{R}^3$, $\mathbf{n} : \Gamma \to \mathbb{R}^3$ *be the unit outer normal along the boundary and* $\mathbf{e} \in \mathbb{L}^2_s(\Omega)$. *Let the space* $\mathbb{M}_0$ *be defined as*

$$\mathbb{M}_0 := \{\mathbf{s} \in \mathbb{L}^2_s(\Omega) \,|\, \mathbf{div}(\mathbf{s}) = \mathbf{0} \textit{ in } \mathbf{H}^{-1}(\Omega),\ \mathbf{sn} = \mathbf{0} \textit{ in } \mathbf{H}^{-1/2}(\Gamma)\}. \tag{2.4.9}$$

There exists a vector field $\mathbf{v} \in \mathbf{H}^1_0(\Omega)$ *such that* $\mathbf{e} = \nabla_s \mathbf{v}$ *if and only if*

$$\int_\Omega \mathbf{e} : \mathbf{s} = 0 \quad \textit{for all } \mathbf{s} \in \mathbb{M}_0, \tag{2.4.10}$$

If (2.4.9) is satisfied then the vector field $\mathbf{v}$ *is unique.*

Donati's characterizations are used for the reformulation of pure traction and pure displacement elasticity problems in an intrinsic way.
In [4] is considered the pure traction problem of three-dimensional linearized elasticity, with the interior forces $\mathbf{f} \in \mathbf{L}^{6/5}(\Omega)$ and the forces on the boundary $\mathbf{g} = \mathbf{0}$. The intrinsic formulation of the problem is based on the space

$$\mathbb{E}_2(\Omega) := \left\{ \mathbf{e} \in \mathbb{L}^2_s(\Omega) \,|\, \int_\Omega \mathbf{e} : \mathbf{s} = 0,\ \forall \mathbf{s} \in \mathbb{H}^1_{0,s}(\Omega) \text{ such that } \mathbf{div}(\mathbf{s}) = \mathbf{0} \text{ in } \mathbf{L}^2(\Omega) \right\}.$$

It is proved (see [4]) that the operator $\nabla_s : \dot{\mathbb{H}}^1(\Omega) \to \mathbb{E}_2(\Omega)$ is bijective and continuous and therefore its inverse operator $\mathcal{F}_2 := \nabla_s^{-1} : \mathbb{E}_2(\Omega) \to \dot{\mathbb{H}}^1(\Omega)$ is an isomorphism and allows the intrinsic reformulation of the pure traction linearized elasticity problem (1.4.7) as:
Find $\mathbf{e} \in \mathbb{E}_2(\Omega)$ *such that*:

$$\int_\Omega \mathbf{A}\mathbf{e} : \tilde{\mathbf{e}} = L \circ \mathcal{F}_2(\tilde{\mathbf{e}}),\ \forall \tilde{\mathbf{e}} \in \mathbb{E}_2(\Omega). \tag{2.4.11}$$

This variational problem is equivalent with the minimization problem:
Find $\mathbf{e} \in \mathbb{E}_2(\Omega)$ *such that*:

$$J_2(\mathbf{e}) = \inf_{\tilde{\mathbf{e}} \in \mathbb{E}_2(\Omega)} J_2(\tilde{\mathbf{e}}),\ \textit{where } J_2(\tilde{\mathbf{e}}) = \frac{1}{2} \int_\Omega \mathbf{A}\tilde{\mathbf{e}} : \tilde{\mathbf{e}} - L \circ \mathcal{F}_2(\tilde{\mathbf{e}}). \tag{2.4.12}$$

The functional L is defined in (1.4.4). Applying the Lax-Milgram Lemma shows that the equivalent problems (2.4.11) and (2.4.12) have an unique solution (see [4]).

Remark 2.4.12. From [4] results that $\mathbb{E}_1$ and $\mathbb{E}_2$ coincides.

In a similar way the pure displacement linearized elasticity problem can be reformulated in an intrinsic way.
In [4], for $\mathbf{f} \in \mathbf{L}^{6/5}(\Omega)$ and $\mathbf{g} = \mathbf{0}$, the variational formulation (1.4.11) of the pure displacement linearized elasticity problem is rewritten in an intrinsic way based on the space:

$$\mathbb{E}_3(\Omega) = \left\{ \mathbf{e} \in \mathbb{L}^2_s(\Omega) \,|\, \int_\Omega \mathbf{e} : \mathbf{s} = 0,\ \forall \mathbf{s} \in \mathbb{L}^2_s(\Omega) \text{ such that } \mathbf{div}(\mathbf{s}) = \mathbf{0} \text{ in } \mathbf{H}^{-1}(\Omega) \right\}.$$

It is proved ([4]) that the operator $\mathcal{F}_3 := \nabla_s^{-1} : \mathbb{E}_3(\Omega) \to \mathbf{H}_0^1(\Omega)$ is an isomorphism. Therefore the weak intrinsic (equivalent) forms of the pure displacement linearized elasticity problem are:
Find $\mathbf{e} \in \mathbb{E}_3(\Omega)$ *such that*:

$$\int_\Omega \mathbf{A}\mathbf{e} : \tilde{\mathbf{e}} = L \circ \mathcal{F}_3(\tilde{\mathbf{e}}),\ \forall \tilde{\mathbf{e}} \in \mathbb{E}_3(\Omega). \tag{2.4.13}$$

Find $\mathbf{e} \in \mathbb{E}_3(\Omega)$ *such that*:

$$J_3(\mathbf{e}) = \inf_{\tilde{\mathbf{e}} \in \mathbb{E}_3(\Omega)} J_3(\tilde{\mathbf{e}}),\ \textit{where}\ J_3(\tilde{\mathbf{e}}) = \frac{1}{2} \int_\Omega \mathbf{A}\tilde{\mathbf{e}} : \tilde{\mathbf{e}} - L \circ \mathcal{F}_3(\tilde{\mathbf{e}}). \tag{2.4.14}$$

2.5 Discretization

The intrinsic discretization of a particular problem is obtained by restricting the intrinsic minimization of the energy functional J to some finite dimensional space. There are two possibilities: to use a Galerkin discretization with these finite elements (see [18]) or to consider the problem as a minimization problem over the finite element space ([4, 26]). According to Theorem 1.1.5 and Remark 1.1.6, these two discretization methods are equivalent. Throughout this thesis, we restrict to piecewise polynomial finite element spaces. As in the classical case (e.g. non intrinsic case) we can obtain conforming and non-conforming element spaces. An intrinsic conforming piecewise constant finite element space was defined for the pure traction elasticity problem in [26, 25], without providing an explicit expression of its basis functions. Here, the intrinsic finite element space is $\mathbb{E}^h \subset \mathbb{E}_1$, with $\mathbb{E}_1$ defined in (2.4.3). The space $\mathbb{E}^h$ is a curl **curl** free edge type finite element space in the sense of Nédélec. The degrees of freedom in the two-dimensional case are given by

$$d_i(\mathbf{e}) = \int_{s_i} \tau_i \cdot \mathbf{e}\tau_i,\ i = 1, 2, 3, \tag{2.5.1}$$

where s_i denote the edges of a non-degenerate triangle T and τ_i denotes a unit vector parallel to s_i. The set $d_i,\ i = 1, 2, 3,$ is $\mathbb{P}_0(T, \mathbb{S}_2)$ - unisolvent. A generalization for a tetrahedron considers the degrees of freedom $d_i,\ i = 1, \ldots, 6$, which are unisolvent in $\mathbb{P}_0(T, \mathbb{S}_3)$. For a given triangulation $\mathcal{T}^h$ the finite element space has the form:

$$\begin{aligned} \tilde{\mathbb{E}}^h = \{&\mathbf{e}^h \in \mathbb{L}_s^2(\Omega); \mathbf{e}^h|_T \in \mathbb{P}_0(T, \mathbb{S}_2),\ \forall T \in \mathcal{T}^h \text{ and} \\ &\int_E \tau \cdot (\mathbf{e}^h|_{T_1})\tau dl = \int_E \tau \cdot (\mathbf{e}^h|_{T_2})\tau dl, \forall\ E = T_1 \cap T_2 \in \mathcal{E}^h, \text{with } T_1, T_2 \in \mathcal{T}^h\}, \end{aligned}$$

where $\mathcal{E}^h$ is the set of interior edges of $\mathcal{T}^h$.
In Chapter 3 we introduce an intrinsic approach to obtain conforming and non-conforming finite element spaces of arbitrary degree for the Poisson equation given in (3.2.1). We use the conforming Galerkin discretization of the variational problem (3.2.6) by the intrinsic finite elements:
Find $\mathbf{e}_\mathcal{T} \in \mathbf{E}_\mathcal{T}^p$ *such that*

$$\int_\Omega \varepsilon \mathbf{e}_\mathcal{T} \cdot \tilde{\mathbf{e}}_\mathcal{T} = \int_\Omega \rho \Lambda \tilde{\mathbf{e}}_\mathcal{T}, \quad \forall \tilde{\mathbf{e}}_\mathcal{T} \in \mathbf{E}_\mathcal{T}^p,$$

where the conforming finite element space is defined as

$$\mathbf{E}_{\mathcal{T}}^{p} := \left\{ \mathbf{e} \in \mathbf{H}^0(\Omega), \text{ such that } \forall \tau \in \mathcal{T} : \mathbf{e}|_{\overset{\circ}{\tau}} \in \mathbf{P}_p \text{ and } \int_\Omega \mathbf{e} \cdot \mathbf{curl} v = 0 \quad \forall v \in H^1(\Omega) \right\}.$$

The main idea is to find triangle-, edge- and vertex-oriented basis functions of the intrinsic finite element space using a local characterization of the finite element space. We obtain that, in the conforming case, these basis functions are the gradients of standard hp-finite element basis functions and all conforming subspaces which are piecewise polynomial are spanned by these basis functions. Using the same reasoning as in the conforming case we derive a non-conforming discretization of the variational problem (3.2.6). All the piecewise polynomial non-conforming subspaces of degree p are spanned by the gradients of standard hp-finite element basis functions enriched by some edge-oriented non-conforming basis functions for p even and by some triangle-supported non-conforming basis functions for p odd. The explicit expression of these basis functions can be found in Section 3.4.2. To our knowledge the non-conforming intrinsic method was not treated in other articles to this extend. In [26] the possibility to obtain non-conforming intrinsic elements was only mentioned.
In Chapter 4 the same idea is extended to the linearized pure traction problem.

2.6 Miscellaneous Remarks

1) An important ingredient to prove the existence and uniqueness of the solution of minimization problems presented in Sections 2.3 and 2.4 is Korn's inequality established in the appropriate spaces. Korn type inequalities play an important role in the theory of intrinsic methods. These kind of inequalities must be established for different function spaces depending on the problem that needs to be solved. They are also useful in proving the existence of different isomorphisms and furthermore lead to a priori error estimates. We give in the following some Korn's type inequalities.
Korn's inequality in $L^2(\Omega)$ ([48]):

$$\|v\|_{L^2(\Omega)} \leq C \left(\|v\|_{H^{-1}(\Omega)} + \|\nabla v\|_{H^{-1}(\Omega)} \right), \ \forall v \in L^2(\Omega). \tag{2.6.1}$$

A matrix analogue of (2.6.1) is Korn's inequality in $\mathbf{L}^2(\Omega)$ ([4]):

$$\|\mathbf{v}\|_{\mathbf{L}^2(\Omega)} \leq C \left(\|\mathbf{v}\|_{\mathbf{H}^{-1}(\Omega)} + \|\nabla_s \mathbf{v}\|_{\mathbf{H}^{-1}(\Omega)} \right), \ \forall \mathbf{v} \in \mathbf{L}^2(\Omega). \tag{2.6.2}$$

Korn's inequalities in $\mathbf{H}_0^1(\Omega)$ and $\mathbf{H}^1(\Omega)$ respectively ([4]) are given by:

$$\|\mathbf{v}\|_{\mathbf{H}^1(\Omega)} \leq C_0 \left(\|\mathbf{v}\|^2_{\mathbf{L}^2(\Omega)} + \|\nabla_s \mathbf{v}\|^2_{\mathbb{L}^2_s(\Omega)} \right)^{\frac{1}{2}}, \qquad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \tag{2.6.3}$$

$$\|\mathbf{v}\|_{\mathbf{H}^1(\Omega)} \leq C \left(\|\mathbf{v}\|^2_{\mathbf{L}^2(\Omega)} + \|\nabla_s \mathbf{v}\|^2_{\mathbb{L}^2_s(\Omega)} \right)^{\frac{1}{2}}, \qquad \forall \mathbf{v} \in \mathbf{H}^1(\Omega). \tag{2.6.4}$$

2) Another issue to be taken into account in the intrinsic formulation is related to the boundary conditions, which usually are given in terms of the classical unknown. One possible solution for the Dirichlet boundary condition is to incorporate it in the discrete intrinsic problem using an efficient approximation $\mathfrak{F}_h$ of the isomorphism $\mathfrak{F}$ and to set the condition $\mathfrak{F}_h(\mathbf{e}_h) = 0$ on the boundary Γ (see [26]). Another possibility is to define the discrete minimization problem using lifting operators $\mathfrak{F}_h$, where the Dirichlet boundary conditions are incorporated ([18]).

3) Based on the equivalence of Poincaré's theorem and Saint Venant's theorem with Lions' Lemma and a matrix analogue of Lions' Lemma, respectively, it is proved in [3] that Saint Venant's Theorem is a "matrix analogue" of Poincaré's theorem. We recall the classical Lions' Lemma and some of its extensions.

Lemma 2.6.1. *(Lions' Lemma [33], [4]) If Ω is a bounded, open subset of $\mathbb{R}^3$ with a smooth boundary and $v \in H^{-1}(\Omega)$ satisfies the condition $\nabla v \in \mathbf{H}^{-1}(\Omega)$ then $v \in L^2(\Omega)$.*

Further extensions of Lions' Lemma were given for domains with Lipschitz continuous boundary and for more general spaces of functions. Lions' Lemma in $\mathbf{H}^m(\Omega)$ is proved in [5]:

Lemma 2.6.2. *Let Ω be a bounded, open subset of $\mathbb{R}^3$ with a smooth boundary. If $\nabla v \in \mathbf{H}^m(\Omega)$ then $v \in H^{m+1}(\Omega)$.*

Another proof of this extension is given by Kevasan (cf. [38]).
A matrix form of Lions' Lemma can be found in [38, 2]:

Lemma 2.6.3. *If Ω is a bounded, open subset of $\mathbb{R}^3$ with a smooth boundary and $\nabla_s \mathbf{v} \in \mathbb{H}^{-1}(\Omega)$ then $\mathbf{v} \in \mathbf{L}^2(\Omega)$.*

An interesting perspective is to extend the results obtained from the vector field theory to the case of matrix fields based on analogies presented in Table 1.

Vector case	*Matrix analogue*
∇	∇_s
curl	curl **curl**
Lions' Lemma in $\mathbf{H}^{-1}(\Omega)$ (Lemma 2.6.2)	Lions' Lemma in $\mathbb{H}_s^{-1}(\Omega)$ (Lemma 2.6.3)
Poincaré's Theorem in $\mathbf{H}^{-1}(\Omega)$ (Theorem 2.4.5)	Saint Venant's Theorem in $\mathbb{H}_s^{-1}(\Omega)$ (Theorem 2.4.6)
Korn's inequality in $L^2(\Omega)$ (2.6.1)	Korn's inequality in $\mathbf{L}^2(\Omega)$ (2.6.2)

Table 1. Correspondences between vector and matrix case.

The existence of a matrix analogue of the results obtained for vector fields gives us the possibility to generalize the intrinsic approach proposed in Chapter 3 for Poisson's equation to the elasticity pure traction problem in Chapter 4.

3

Intrinsic Finite Element Methods for the Computation of Fluxes for Poisson's Equation*

3.1 Introduction

The goal of this chapter is to develop a general method for the *derivation* of *intrinsic* conforming and non-conforming finite elements from theoretical principles for the discretization of elliptic partial differential equations. In this chapter we consider an intrinsic approach for the direct computation of the fluxes for problems in potential theory. We develop a general method for the derivation of intrinsic conforming and non-conforming finite element spaces and appropriate lifting operators for the evaluation of the right-hand side from abstract theoretical principles related to the second Strang Lemma. The idea of our general approach is applied in Chapter 4 to obtain intrinsic finite element spaces for elasticity problems.
We derive piecewise polynomial intrinsic conforming finite element spaces of any degree p and give an explicit form for a local basis. Then, we employ the stability and convergence theory for non-conforming finite elements based on the second Strang Lemma and derive from these principles weak compatibility conditions for non-conforming finite elements across simplex boundary. In other words, we show that local polynomial finite element spaces for elliptic problems in divergence form *must* satisfy those compatibility conditions in order to estimate the perturbation in the second Strang Lemma in a consistent way.
The convergence of the proposed intrinsic finite element method is proved.
As a simple model problem for the introduction of our method, we consider Poisson's equation but emphasize that this method is applicable also for much more general (systems of) elliptic equations. We consider the intrinsic formulation of Poisson's equation, i.e., the minimization of the energy functional in the space of *admissible* energies which will be defined below. The goal is to construct piecewise polynomial finite element spaces for the *direct* approximation of the physical quantity of interest, i.e., the flux, the electrostatic field, the velocity field, etc. depending on the underlying application. To take into account essential boundary conditions we have to construct a *lifting operator* as the left inverse of the elementwise gradient operator, that is, an operator defined element by element – whose realization turns out to be quite simple.

*This chapter of the thesis is a slightly extended and modified version of [18].

There is a vast literature on various conforming and non-conforming, primal, dual, mixed formulations of elliptic differential equations and conforming as well as non-conforming discretization. Since our main focus is the development of a *concept* for deriving conforming and non-conforming intrinsic finite elements from theoretical principles and not the presentation of a specific new finite element space we omit an extensive list of references on the analysis of specific families of finite elements spaces but refer to the classical monographs [23], [53], and [13], and the references therein.
An intrinsic finite element space for approximating linearized elasticity problems has been developed in [25] and [26] by modifying the lowest order Nédélec finite elements (cf. [46], [47]) such that the compatibility conditions which arise from the intrinsic formulation are satisfied. Intrinsic formulations of the Lamé equations modelling linear three-dimensional elasticity have been first derived in [24].
The approach we propose allows us to recover the non-conforming Crouzeix-Raviart element [30], the Fortin-Soulie element [36], the Crouzeix-Falk element [29], and the Gauss-Legendre elements [8], [7] as well as the standard conforming hp-finite elements.
We underline that the proposed intrinsic method is different from the mixed methods ([9, 11, 13, 42, 47, 46]). Our purpose is to obtain a pure formulation of the problem in the flux variable.
The chapter is organized as follows.
In Section 3.2 we introduce our model problem and the relevant function spaces for the intrinsic formulation of the continuous problem as an energy minimization problem.
In Section 3.3 we derive weak continuity conditions for the characterization of the admissible energy space. Based on these conditions we derive conforming intrinsic polynomial finite element spaces and show that they are (necessarily) the gradients of the well-known Lagrange hp-finite element spaces.
In Section 3.4 we infer from the proof of the second Strang lemma appropriate compatibility conditions at the interfaces between elements of the mesh so that the non-conforming perturbation of the original bilinear form can be estimated in a consistent way. We derive *all* types of piecewise polynomial finite element that satisfy this condition and also derive a local basis for these spaces.
Finally, in Section 3.5 we summarize the main results and give some conclusions.

3.2 Model Problem

We consider the model problem of finding, for a given electric charge density $\rho \in L^2(\Omega)$, an electrostatic field $\mathbf{e}$ in a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, which satisfies

$$-\operatorname{div}(\varepsilon \mathbf{e}) = \rho \qquad \text{in } \Omega, \tag{3.2.1}$$

where ε denotes the electrostatic permeability. In the electrostatic case, one may further write $\mathbf{e} = \nabla\phi$, where ϕ is the electrostatic potential, known up to a constant. We consider that the potential ϕ is constant on each connected component of the boundary $\Gamma := \partial\Omega$. Classically, this amounts to saying that (3.2.1) is complemented with a perfect conductor boundary condition, namely†, $\mathbf{e} \times \mathbf{n}_{|\partial\Omega} = 0$, where $\mathbf{n}$ is the unit outward normal vector field to $\partial\Omega$.

†For $d = 3$, $\mathbf{a} \times \mathbf{b}$ is the usual vector product and in two dimensions we use $\mathbf{a} \times \mathbf{b} := a_1 b_2 - a_2 b_1$.

Throughout the paper we assume that

$$\Omega \subset \mathbb{R}^d \text{ is a bounded Lipschitz domain with connected boundary } \Gamma. \tag{3.2.2}$$

As a consequence of this assumption, $\phi_{|\partial\Omega}$ is constant. Since ϕ is known up to a constant, we may choose an electrostatic potential such that $\phi_{|\partial\Omega} = 0$.
Consequently, the classical variational formulation of the problem is:
Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \varepsilon \nabla u \nabla v = \int_\Omega f v \quad \forall v \in H_0^1(\Omega). \tag{3.2.3}$$

Hence, the variational formulation of (3.2.1) restricted to the domain Ω is based on the space

$$\mathbf{E}(\Omega) := \nabla H_0^1(\Omega).$$

We remember that vector fields and spaces of vector fields are denoting using boldface letters.

Remark 3.2.1. If $\partial\Omega$ consists of disjoint connected components Γ_k, $0 \le k \le q$, i.e., $\partial\Omega = \bigcup_{k=0}^{q} \Gamma_k$, with $\overline{\Gamma_k} \cap \overline{\Gamma_{k'}} = \emptyset$ for $k \neq k'$, then the space $\mathbf{E}(\Omega)$ is given by

$$\mathbf{E}(\Omega) = \left\{ \nabla v \mid v \in H^1(\Omega),\ v\big|_{\Gamma_0} = 0 \quad \text{and, for all } 1 \le k \le q, \quad v\big|_{\Gamma_k} = c_k \right\}$$

for arbitrary constants $c_k \in R$, $1 \le k \le q$. To reduce technicalities in this paper, we will only consider domains that satisfy (3.2.2).

Given a scalar field v, we define its (weak) vector curl using (2.2.8). Likewise, given a vector field $\mathbf{e}$, we define its (weak) scalar curl by (2.2.7).
We recall a well-known result below. The proof is based on Poincaré's theorem and can be found in [43].

Proposition 3.2.2. *Let $\Omega \subset \mathbb{R}^d$ satisfy (3.2.2). The operator $\nabla : H_0^1(\Omega) \to \mathbf{E}(\Omega)$ is an isomorphism and thus its inverse operator $\Lambda : \mathbf{E}(\Omega) \to H_0^1(\Omega)$ is continuous.*
Let $d = 2$. It holds

$$\begin{aligned} \mathbf{E}(\Omega) &= \left\{ \mathbf{e} \in \mathbf{L}^2(\Omega) \mid \int_\Omega \mathbf{e} \cdot \mathbf{curl} v = 0 \quad \forall v \in H^1(\Omega) \right\} \\ &= \left\{ \mathbf{e} \in \mathbf{L}^2(\Omega) \mid \operatorname{curl} \mathbf{e} = 0 \text{ in } H^{-1}(\Omega) \ \textit{and } \mathbf{e} \times \mathbf{n} = 0 \text{ in } H^{-1/2}(\Gamma) \right\}. \end{aligned} \tag{3.2.4}$$

In order to ensure existence and uniqueness of the variational formulation and convergence estimates for the finite element discretization we impose the following assumptions on the electrostatic permeability.

Assumption 3.2.3. *The electrostatic permeability ε in (3.2.1) satisfies $\varepsilon \in L^\infty(\Omega)$ and*

$$0 < \varepsilon_{\min} := \operatorname*{ess\,inf}_{x \in \Omega} \varepsilon(x) \le \operatorname*{ess\,sup}_{x \in \Omega} \varepsilon(x) =: \varepsilon_{\max} < \infty. \tag{3.2.5}$$

There exists a partition $\mathcal{P} := (\Omega_j)_{j=1}^J$ of Ω into J (possibly curved) polygons such that, for all $r \in \mathbb{N}$, it holds

$$\|\varepsilon\|_{PW^{r,\infty}(\Omega)} := \max_{1 \le j \le J} \left\| \varepsilon|_{\Omega_j} \right\|_{W^{r,\infty}(\Omega_j)} < \infty.$$

The intrinsic variational problem reads: Find $\mathbf{e} \in \mathbf{E}(\Omega)$ such that

$$\int_\Omega \varepsilon \mathbf{e} \cdot \tilde{\mathbf{e}} = \int_\Omega \rho \Lambda \tilde{\mathbf{e}} \qquad \forall \tilde{\mathbf{e}} \in \mathbf{E}(\Omega). \tag{3.2.6}$$

Equivalently the solution $\mathbf{e}$ can be characterized as the minimizer on $\mathbf{E}(\Omega)$ of the functional

$$j : \mathbf{E}(\Omega) \to \mathbb{R} \quad j(\tilde{\mathbf{e}}) := \frac{1}{2} \int_\Omega \varepsilon \tilde{\mathbf{e}} \cdot \tilde{\mathbf{e}} - \int_\Omega \rho \Lambda \tilde{\mathbf{e}}. \tag{3.2.7}$$

In most physical applications the quantity $\mathbf{e}$, or the flux $\varepsilon \mathbf{e}$, is the physical quantity of interest rather than the potential $u = \Lambda \mathbf{e}$ and our goal is to *derive* conforming and non-conforming finite element spaces for the direct approximation of $\mathbf{e}$ in (3.2.6) from conditions which arise from the abstract convergence theory.

3.3 Conforming Intrinsic Finite Element Spaces

In this paper we restrict our studies to two-dimensional, bounded, polygonal domains $\Omega \subset \mathbb{R}^2$ and simplicial triangulations $\mathcal{T}$, defined using the conventions from Section 2.2.2.
For $p \in \mathbb{N}_0$ let $\mathcal{P}_p$ denote the space of polynomials of degree $\leq p$, i.e., consisting of the functions $\sum_{i=0}^{p} \sum_{j=0}^{p-i} a_{i,j} x_1^i x_2^j$ for some real coefficients $a_{i,j}$. For $\omega \subset \Omega$, we write $\mathcal{P}_p(\omega)$ for polynomials of degree $\leq p$ defined on ω. Given $\mathcal{T}$, we define the finite element spaces

$$\left.\begin{aligned} S_{\mathcal{T}}^{p,m} &:= \left\{ u \in H^{m+1}(\Omega) \mid \forall \tau \in \mathcal{T} : u|_{\overset{\circ}{\tau}} \in \mathbb{P}_p \right\}, \\ \mathbf{S}_{\mathcal{T}}^{p,m} &:= S_{\mathcal{T}}^{p,m} \times S_{\mathcal{T}}^{p,m}, \end{aligned}\right\} \quad \text{for } m = -1, 0,$$

$$S_{\mathcal{T},0}^{p,m} := S_{\mathcal{T}}^{p,m} \cap H_0^1(\Omega),$$

and

$$\mathbf{E}_{\mathcal{T}}^{p} := \left\{ \mathbf{e} \in \mathbf{S}_{\mathcal{T}}^{p,-1} \mid \int_\Omega \mathbf{e} \cdot \mathbf{curl} v = 0 \quad \forall v \in H^1(\Omega) \right\}. \tag{3.3.1}$$

From (3.2.4) we conclude that $\mathbf{E}_{\mathcal{T}}^{p} \subset \mathbf{E}(\Omega)$ is a piecewise polynomial finite element space which gives rise to the conforming Galerkin discretization of (3.2.6) by these *intrinsic* finite elements: *Find* $\mathbf{e}_{\mathcal{T}} \in \mathbf{E}_{\mathcal{T}}^{p}$ *such that*

$$\int_\Omega \varepsilon \mathbf{e}_{\mathcal{T}} \cdot \tilde{\mathbf{e}}_{\mathcal{T}} = \int_\Omega \rho \Lambda \tilde{\mathbf{e}}_{\mathcal{T}} \qquad \forall \tilde{\mathbf{e}}_{\mathcal{T}} \in \mathbf{E}_{\mathcal{T}}^{p}. \tag{3.3.2}$$

In the rest of Section 3.3, we will derive a local basis for $\mathbf{E}_{\mathcal{T}}^{p}$ and a realization of the lifting operator Λ.

3.3.1 Local Characterization of Conforming Intrinsic Finite Elements

In this section, we will develop a local characterization of conforming intrinsic finite elements. This approach generalizes that of [25], where such finite element approximations were considered for the first time (for the system of two-dimensional linearized elasticity).
For an edge $E \in \mathcal{E} \cup \mathcal{E}_{\partial\Omega}$ let $\mathbf{n}_E$ denote a unit vector which is orthogonal to E. The orientation for the inner edges is arbitrary but fixed while the orientation for the boundary edges is such that $\mathbf{n}_E$ points toward the exterior of Ω. Let $\mathbf{t}_E$ denote an oriented unit vector along E, which obeys the convention that $\det[\mathbf{t}_E, \mathbf{n}_E] = 1$.

For the inner edges $E \in \mathcal{E}$, we define the pointwise tangential jumps $[\mathbf{e} \cdot \mathbf{t}_E]_E : E \to \mathbb{R}$ for $\mathbf{x} \in \overset{\circ}{E}$ by

$$[\mathbf{e} \cdot \mathbf{t}_E]_E (\mathbf{x}) = \lim_{\varepsilon \searrow 0} (\mathbf{e}(\mathbf{x} + \varepsilon \mathbf{n}_E) \cdot \mathbf{t}_E - \mathbf{e}(\mathbf{x} - \varepsilon \mathbf{n}_E) \cdot \mathbf{t}_E).$$

Lemma 3.3.1. *Let the boundary of Ω be connected. The space $\mathbf{E}^p_{\mathcal{T}}$ can be characterized by local conditions according to*

$$\begin{aligned} \mathbf{E}^p_{\mathcal{T}} = \Big\{ \mathbf{e} \in \mathbf{S}^{p,-1}_{\mathcal{T}} \mid & \operatorname{curl}_{\mathcal{T}} \mathbf{e} = 0 \\ & \text{and for all } E \in \mathcal{E} \quad [\mathbf{e} \cdot \mathbf{t}_E]_E = 0 \\ & \text{and for all } E \in \mathcal{E}_{\partial\Omega} \quad \mathbf{e} \cdot \mathbf{t}_E|_E = 0 \Big\}. \end{aligned} \tag{3.3.3}$$

Proof. We denote the right-hand side in (3.3.3) by $\tilde{\mathbf{E}}^p_{\mathcal{T}}$ and prove $\mathbf{E}^p_{\mathcal{T}} = \tilde{\mathbf{E}}^p_{\mathcal{T}}$. in Part a -Part c we prove that $\mathbf{E}^p_{\mathcal{T}} \subseteq \tilde{\mathbf{E}}^p_{\mathcal{T}}$. Let $\mathbf{e} \in \mathbf{E}^p_{\mathcal{T}}$. Consider the curl-condition (3.3.1) with test-fields v.

Part a: For $\tau \in \mathcal{T}$, let $v \in \mathcal{D}(\tau) := \{u \in C^\infty(\tau) \mid \operatorname{supp} u \subset\subset \tau\}$. Then,

$$\int_\tau (\operatorname{curl} \mathbf{e})\, v = \int_\tau \mathbf{e} \cdot \mathbf{curl} v = 0.$$

Since $\tau \in \mathcal{T}$ and $v \in \mathcal{D}(\tau)$ are arbitrary, we conclude that $\operatorname{curl}_{\mathcal{T}} \mathbf{e} = 0$ holds.

Part b: For $E \in \mathcal{E}$, let $\tau_1, \tau_2 \in \mathcal{T}$ be such that $E = \tau_1 \cap \tau_2$. We set $\omega_E := \tau_1 \cup \tau_2$ (cf. (2.2.11)). We choose $v \in \mathcal{D}\left(\overset{\circ}{\omega}_E\right)$. Then

$$\int_{\tau_1} \mathbf{e} \cdot \mathbf{curl} v + \int_{\tau_2} \mathbf{e} \cdot \mathbf{curl} v = 0.$$

For $i = 1, 2$, denote by $\mathbf{n}^i = \left(n^i_1, n^i_2\right)^{\mathrm{T}}$ the exterior normal for τ_i. Trianglewise partial integration yields (by taking into account $v = 0$ on $\partial\omega_E$)

$$\begin{aligned} 0 &= \int_{\partial\tau_1} \left(e_1 n^1_2 - e_2 n^1_1\right) v + \int_{\partial\tau_2} \left(e_1 n^2_2 - e_2 n^2_1\right) v - \int_{\omega_E} (\operatorname{curl}_{\mathcal{T}} \mathbf{e})\, v \\ &= \int_E \left(e_1 n^1_2 - e_2 n^1_1\right) v + \int_E \left(e_1 n^2_2 - e_2 n^2_1\right) v - \int_{\omega_E} (\operatorname{curl}_{\mathcal{T}} \mathbf{e})\, v. \end{aligned}$$

We already proved $\operatorname{curl}_{\mathcal{T}} \mathbf{e} = 0$. Note that $\left(-n^1_2, n^1_1\right)^{\mathrm{T}} = -\left(-n^2_2, n^2_1\right)^{\mathrm{T}}$ is tangential to E so that

$$0 = \int_E [\mathbf{e} \cdot \mathbf{t}_E]_E\, v.$$

Since $v \in \mathcal{D}\left(\overset{\circ}{\omega}_E\right)$ is arbitrary, we conclude $[\mathbf{e} \cdot \mathbf{t}_E]_E = 0$.

Part c: Let $E \in \mathcal{E}_{\partial\Omega}$ and $\tau \in \mathcal{T}$ such that $E \subset \partial\tau$. Let

$$\mathcal{D}_E(\tau) := \left\{ v|_\tau : v \in \mathcal{D}\left(\mathbb{R}^2\right) \text{ and } v = 0 \text{ in some neighborhood of } \Omega \backslash \tau \right\}.$$

Repeating the argument as in Part b by taking into account that $v \in \mathcal{D}_E(\tau)$ in general does not vanish on E leads to $\mathbf{e} \cdot \mathbf{t}_E = 0$ in this case.

Thus, we have proved $\mathbf{E}^p_{\mathcal{T}} \subset \tilde{\mathbf{E}}^p_{\mathcal{T}}$.

Part d: To prove the opposite inclusion we consider $\mathbf{e} \in \tilde{\mathbf{E}}_{\mathcal{T}}^{p}$. Then, for all $v \in H^1(\Omega)$ it holds

$$\begin{aligned}
{}_{(H^1(\Omega))'}\langle \operatorname{curl}\mathbf{e}, v\rangle_{H^1(\Omega)} &= \int_\Omega \mathbf{e}\cdot\mathbf{curl}v = \sum_{\tau\in\mathcal{T}}\int_\tau \mathbf{e}\cdot\mathbf{curl}v \\
&= \sum_{\tau\in\mathcal{T}}\int_\tau (\operatorname{curl}_{\mathcal{T}}\mathbf{e})\,v + \sum_{\tau\in\mathcal{T}}\int_{\partial\tau}\left(-e_1 n_2^\tau + e_2 n_1^\tau\right)v \\
&= \sum_{\tau\in\mathcal{T}}\int_\tau (\operatorname{curl}_{\mathcal{T}}\mathbf{e})\,v + (-1)^{\sigma_E}\sum_{E\in\mathcal{E}}\int_E [\mathbf{e}\cdot\mathbf{t}_E]_E\,v \\
&\qquad + \sum_{E\in\mathcal{E}_{\partial\Omega}}\int_E (\mathbf{e}\cdot\mathbf{t}_E)\,v \\
&= 0.
\end{aligned}$$

Above, $\sigma_E \in \{0,1\}$, depending on the orientation of $\mathbf{t}_E$.
Hence, $\tilde{\mathbf{E}}_{\mathcal{T}}^{p} \subset \mathbf{E}_{\mathcal{T}}^{p}$ and the assertion follows. □

3.3.2 Integration

We start with a lemma on integration of curl-free polynomials. Let

$$\mathbf{P}_{\text{curl}}^{p} := \{\mathbf{e}\in\mathcal{P}_p\times\mathcal{P}_p : \operatorname{curl}\mathbf{e} = 0\} \tag{3.3.4}$$

and, for $\tau\in\mathcal{T}$, we write $\mathbf{P}_{\text{curl}}^{p}(\tau) := \left\{\mathbf{e}|_\tau : \mathbf{e}\in\mathbf{P}_{\text{curl}}^{p}\right\}$ to indicate the domain of the functions explicitly.

Lemma 3.3.2. *For any $\tau\in\mathcal{T}$ and any $\mathbf{e}\in\mathbf{P}_{\text{curl}}^{p}(\tau)$, it holds*

$$\emptyset \neq \left\{u\in H^1(\tau) \mid \nabla u = \mathbf{e}\right\} \subset \mathcal{P}_{p+1}(\tau)\,. \tag{3.3.5}$$

Proof. Let $\tau\in\mathcal{T}$ and $\mathbf{e}\in\mathbf{P}_{\text{curl}}^{p}(\tau)$. In [43, 5] it is proved that there exists $u\in H^1(\tau)$, unique up to a constant, such that $\nabla u = \mathbf{e}$ and, hence, the left-hand side in (3.3.5) is proved. Let $\mathbf{m}_\tau$ be the center of mass for τ and let $\gamma_{\mathbf{x}}$ denoting the straight path $\overline{\mathbf{m}_\tau\mathbf{x}}$ (cf. Figure 3.1). Then Poincaré's theorem yields that the path integral

$$U(\mathbf{x}) := \int_{\gamma_{\mathbf{x}}} \mathbf{e} \tag{3.3.6}$$

defines some U such that $\nabla U = \mathbf{e}$. Since $\mathbf{e}\in\mathcal{P}_{\text{curl}}^{p}(\tau)$, there are coefficients $\mathbf{a}_\mu\in\mathbb{R}^2$ such that

$$\mathbf{e}(\mathbf{x}) = \sum_{|\mu|\le p}\mathbf{a}_\mu(\mathbf{x}-\mathbf{m}_\tau)^\mu$$

with the usual multiindex notation $\mu\in\mathbb{N}_0^2$, $|\mu| := \mu_1+\mu_2$, $\mathbf{w}^\mu := w_1^{\mu_1}w_2^{\mu_2}$. To evaluate the integral in (3.3.6) we employ the affine pullback $\chi_{\mathbf{x}} : [0,1]\to\overline{\mathbf{m}_\tau\mathbf{x}}$, $\chi_{\mathbf{x}} := \mathbf{m}_\tau + t(\mathbf{x}-\mathbf{m}_\tau)$

and obtain

$$\begin{aligned} U(\mathbf{x}) &= \int_0^1 \mathbf{e} \circ \chi_{\mathbf{x}}(t) \cdot \chi'_{\mathbf{x}}(t)\, dt \\ &= \sum_{|\mu|\le p} \mathbf{a}_\mu \cdot (\mathbf{x} - \mathbf{m}_\tau) \int_0^1 (t(\mathbf{x} - \mathbf{m}_\tau))^\mu\, dt \\ &= \sum_{|\mu|\le p} (\mathbf{a}_\mu \cdot (\mathbf{x} - \mathbf{m}_\tau))(\mathbf{x} - \mathbf{m}_\tau)^\mu \int_0^1 t^{|\mu|} dt \\ &= \sum_{|\mu|\le p} \mathbf{a}_\mu \cdot (\mathbf{x} - \mathbf{m}_\tau) \frac{(\mathbf{x} - \mathbf{m}_\tau)^\mu}{|\mu|+1} \in \mathbb{P}_{p+1}. \end{aligned}$$

Since the functions in the set $\{\ldots\}$ in (3.3.5) differ only by a constant we have proved the second inclusion in (3.3.5). $\square$

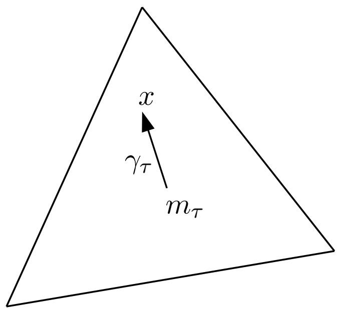

Figure 3.1: *The straight path $\gamma_{\mathbf{x}}$*

Lemma 3.3.2 motivates the definition of the local lifting $\lambda_\tau^c : \mathbf{P}^p_{\mathrm{curl}}(\tau) \to \mathcal{P}^{p+1}(\tau)$ for $\tau \in \mathcal{T}$, $\mathbf{e} \in \mathbf{P}^p_{\mathrm{curl}}(\tau)$, and $c \in \mathbb{R}$ by

$$\lambda_\tau^c(\mathbf{e}) := U + c \quad \text{with} \quad U \text{ as in (3.3.6)}. \tag{3.3.7}$$

Note that the space in (3.3.5) satisfies

$$\left\{u \in H^1(\tau) \mid \nabla u = \mathbf{e}\right\} = \{\lambda_\tau^c(\mathbf{e}) : c \in \mathbb{R}\}.$$

Proposition 3.3.3. *Let the boundary of Ω be connected. $\Lambda : \mathbf{E}^p_{\mathcal{T}} \to S^{p+1,0}_{\mathcal{T},0}$ is an isomorphism with inverse $\nabla : S^{p+1,0}_{\mathcal{T},0} \to \mathbf{E}^p_{\mathcal{T}}$.*

Proof. From Lemma 3.3.2 we conclude that

$$\Lambda \mathbf{E}^p_{\mathcal{T}} \subset S^{p+1,-1}_{\mathcal{T}}$$

holds. Since $\mathbf{E}^p_{\mathcal{T}} \subset \mathbf{E}$, the mapping properties of the lifting Λ imply

$$\Lambda \mathbf{E}^p_{\mathcal{T}} \subset H^1_0(\Omega).$$

Hence

$$\Lambda \mathbf{E}^p_{\mathcal{T}} \subset S^{p+1,-1}_{\mathcal{T}} \cap H^1_0(\Omega) = S^{p+1,0}_{\mathcal{T},0}.$$

On the other hand, we have $S_{\mathcal{T},0}^{p+1,0} \subset H_0^1(\Omega)$ and hence $\nabla S_{\mathcal{T},0}^{p+1,0} \subset \mathbf{E}$. Furthermore, it is clear that

$$\nabla S_{\mathcal{T},0}^{p+1,0} \subset \mathbf{S}_{\mathcal{T}}^{p,-1}.$$

Hence,

$$\nabla S_{\mathcal{T},0}^{p+1,0} \subset \mathbf{S}_{\mathcal{T}}^{p,-1} \cap \mathbf{E} = \mathbf{E}_{\mathcal{T}}^{p}$$

from which we finally conclude that

$$S_{\mathcal{T},0}^{p+1,0} \subset \Lambda \mathbf{E}_{\mathcal{T}}^{p}$$

holds which completes the proof. □

3.3.3 A Local Basis for Conforming Intrinsic Finite Elements

Proposition 3.3.3 shows that a basis for the intrinsic finite element space $\mathbf{E}_{\mathcal{T}}^{p}$, can easily be constructed by using the standard basis functions for *hp*-finite element spaces given in Section 2.2.2. The Lagrange basis for $S_{\mathcal{T},0}^{p,0}$ can be indexed by the nodal points $N \in \mathcal{N}^p$ and is characterized by

$$b_{p,N}^{\mathcal{T}} \in S_{\mathcal{T},0}^{p,0} \quad \text{and} \quad \forall N' \in \mathcal{N}^p \quad b_{p,N}^{\mathcal{T}}\left(N'\right) = \begin{cases} 1 & N = N', \\ 0 & N \neq N'. \end{cases} \tag{3.3.8}$$

Recall that the set $\mathcal{N}^p$ is defined in 2.2.16 and the triangles in $\mathcal{T}$ and the edges in $\mathcal{E}$ are by convention closed.
We define the following subspaces of $\mathbf{E}_{\mathcal{T}}^{p}$:

$$\mathbf{B}_{\tau}^{p} := span\left\{\nabla b_{p+1,N}^{\mathcal{T}} \mid N \in \mathring{\tau} \cap \mathcal{N}^{p+1}\right\} \text{ for all } \tau \in \mathcal{T}, \tag{3.3.9}$$

$$\mathbf{B}_{E}^{p} := span\left\{\nabla b_{p+1,N}^{\mathcal{T}} \mid N \in \mathring{E} \cap \mathcal{N}^{p+1}\right\} \text{ for all } E \in \mathcal{E}, \tag{3.3.10}$$

$$\mathbf{B}_{V}^{p} := span\left\{\nabla b_{p+1,V}^{\mathcal{T}}\right\} \text{ for all } V \in \mathcal{V}. \tag{3.3.11}$$

Proposition 3.3.4. *Let the boundary of Ω be connected. The space $\mathbf{E}_{\mathcal{T}}^{p}$ can be decomposed as the direct sum*

$$\mathbf{E}_{\mathcal{T}}^{p} = \left(\bigoplus_{V\in\mathcal{V}} \mathbf{B}_{V}^{p}\right) \oplus \left(\bigoplus_{E\in\mathcal{E}} \mathbf{B}_{E}^{p}\right) \oplus \left(\bigoplus_{\tau\in\mathcal{T}} \mathbf{B}_{\tau}^{p}\right). \tag{3.3.12}$$

Proof. Proposition 3.3.3 implies that $(\nabla b_{p+1,N}^{\mathcal{T}})_{N\in\mathcal{N}^{p+1}}$ is a basis of $\mathbf{E}_{\mathcal{T}}^{p}$. The assertion follows simply by sorting these basis functions, according as to whether they are associated with a single triangle, with two triangles with a side in common, and with triangles with a vertex in common. □

Corollary 3.3.5. *The subspaces defined in (3.3.9), (3.3.10), (3.3.11) are triangle-, edge-, and vertex-oriented local subspaces of $\mathbf{E}_{\mathcal{T}}^{p}$ and can be expressed as follows:*
The triangle- oriented subspace $\mathbf{B}_{\tau}^{p}$ is given by:

$$\mathbf{B}_{\tau}^{p} = \left\{\mathbf{e} \in \mathbf{E}_{\mathcal{T}}^{p} \mid \operatorname{supp} \mathbf{e} \subset \tau\right\}. \tag{3.3.13}$$

The edge-oriented subspace $\mathbf{B}_{E}^{p}$ will be constructed such that the following direct sum decomposition holds

$$\mathbf{B}_{E}^{p} \oplus \left(\bigoplus_{\tau\in\mathcal{T}_E} \mathbf{B}_{\tau}^{p}\right) = \left\{\mathbf{e} \in \mathbf{E}_{\mathcal{T}}^{p} \mid \operatorname{supp} \mathbf{e} \subset \omega_E\right\}. \tag{3.3.14}$$

The vertex-oriented subspace $\mathbf{B}_V^p$ *will be constructed such that the following condition is satisfied*

$$\mathbf{B}_V^p \oplus \left(\bigoplus_{E \in \mathcal{E}_V} \mathbf{B}_E^p \right) \oplus \left(\bigoplus_{\tau \in \mathcal{T}_V} B_\tau^p \right) = \left\{ \mathbf{e} \in \mathbf{E}_{\mathcal{T}}^p \mid \operatorname{supp} \mathbf{e} \subset \omega_V \right\}. \tag{3.3.15}$$

Remark 3.3.6. Corollary 3.3.3 and the definition of triangle-, edge-, and vertex-oriented local subspaces of $\mathbf{E}_{\mathcal{T}}^p$ shows that (3.3.2) is equivalent to the standard Galerkin finite element formulation of (3.2.1): *Find* $u_{\mathcal{T}} \in S_{\mathcal{T},0}^{p+1,0}$ such that

$$\int_\Omega \varepsilon \nabla u_{\mathcal{T}} \cdot \nabla v_{\mathcal{T}} = \int_\Omega \rho v_{\mathcal{T}} \qquad \forall v_{\mathcal{T}} \in S_{\mathcal{T},0}^{p+1,0}$$

via $e_{\mathcal{T}} = \nabla u_{\mathcal{T}}$. However, the derivation via the intrinsic variational formulation has the advantage of providing insights on how to design non-conforming intrinsic finite element.

3.4 Non-Conforming Intrinsic Finite Elements

3.4.1 (Implicit) Definition of Non-Conforming Intrinsic Finite Elements

In this section, we will define non-conforming intrinsic finite element spaces to approximate the solution of (3.2.6). As a minimal requirement we assume that the non-conforming finite element space $\mathbf{E}_{\mathcal{T},\mathrm{nc}}^p$ satisfies

$$\mathbf{E}_{\mathcal{T},\mathrm{nc}}^p \subset \mathbf{L}^2(\Omega) \quad \text{and} \quad \mathbf{E}_{\mathcal{T},\mathrm{nc}}^p \not\subset \mathbf{E}(\Omega) \quad \text{and} \quad \dim \mathbf{E}_{\mathcal{T},\mathrm{nc}}^p < \infty. \tag{3.4.1}$$

We further require that $\mathbf{E}_{\mathcal{T},\mathrm{nc}}^p$ is a piecewise polynomial, trianglewise curl-free finite element space and that the conforming space $\mathbf{E}_{\mathcal{T}}^p$ is a subspace of $\mathbf{E}_{\mathcal{T},\mathrm{nc}}^p$:

$$\mathbf{E}_{\mathcal{T}}^p \subset \mathbf{E}_{\mathcal{T},\mathrm{nc}}^p \subset \left\{ \mathbf{e} \in \mathbf{S}_{\mathcal{T}}^{p,-1} \mid \operatorname{curl}_{\mathcal{T}} \mathbf{e} = 0 \right\}. \tag{3.4.2}$$

For the definition of a variational formulation we have to extend the lifting operator Λ to an operator $\Lambda_{\mathcal{T}}$ which satisfies

$$\Lambda_{\mathcal{T}} : \left(\mathbf{E}_{\mathcal{T},\mathrm{nc}}^p + \mathbf{E}(\Omega) \right) \to L^2(\Omega) \tag{3.4.3}$$

$$\Lambda_{\mathcal{T}} : \mathbf{E}_{\mathcal{T},\mathrm{nc}}^p \to S_{\mathcal{T}}^{p+1,-1} \tag{3.4.4}$$

as well as the consistency condition

$$\Lambda_{\mathcal{T}} \mathbf{e} = \Lambda \mathbf{e} \qquad \forall \mathbf{e} \in \mathbf{E}(\Omega). \tag{3.4.5}$$

The complete definitions of $\mathbf{E}_{\mathcal{T},\mathrm{nc}}^p$ and $\Lambda_{\mathcal{T}}$ will be based on the convergence theory for non-conforming finite elements according to the second Strang Lemma (cf. [23, Th. 4.2.2]): this lemma will specify how to define them and obtain in the end an optimal order of convergence (see Theorem 3.4.5 hereafter).
In the same spirit as in Section 3.3, we first define the operator $\Lambda_{\mathcal{T}}$ elementwise by the local lifting operators $\lambda_\tau^{c_\tau}$ as in (3.3.7):

$$\left. (\Lambda_{\mathcal{T}} \mathbf{e}) \right|_{\mathring{\tau}} := \lambda_\tau^{c_\tau} \left(\left. \mathbf{e} \right|_{\mathring{\tau}} \right) \qquad \forall \tau \in \mathcal{T} \quad \forall \mathbf{e} \in \mathbf{E}_{\mathcal{T},\mathrm{nc}}^p. \tag{3.4.6}$$

Note that the coefficients $(c_\tau)_{\tau\in\mathcal{T}}$ are at our disposal.
From (3.4.6) we conclude that $\nabla_\mathcal{T}$ is a left-inverse to $\Lambda_\mathcal{T}$, i.e.,

$$\forall \mathbf{e} \in \mathbf{E}^p_{\mathcal{T},\mathrm{nc}} : \nabla_\mathcal{T}\Lambda_\mathcal{T}\mathbf{e} = \mathbf{e}. \tag{3.4.7}$$

A compatibility assumption on $\mathbf{E}^p_{\mathcal{T},\mathrm{nc}}$ concerning the jumps of functions across edges is formulated next. For an edge E with endpoints A^E, B^E the affine mapping $\chi_E : [-1,1] \to E$ is given by $\chi_E(\xi) = A^E + \frac{\xi+1}{2}\left(B^E - A^E\right)$. The space of univariate polynomials of degree $\leq p$ along the edge E is given by

$$\mathcal{P}_p(E) := \left\{ q \circ \chi_E^{-1} \mid q \text{ is a polynomial of degree } \leq p \text{ on } [-1,1] \right\}. \tag{3.4.8}$$

On the one hand, given $\mathbf{e} \in \mathbf{E}^p_\mathcal{T}$, one has $[\Lambda_\mathcal{T}\mathbf{e}]_E = 0$ for all $E \in \mathcal{E}$, and $\Lambda_\mathcal{T}\mathbf{e} = 0$ on $\partial\Omega$. On the other hand, for elements of the non-conforming finite element space $\mathbf{E}^p_{\mathcal{T},\mathrm{nc}}$, we require that these conditions are *weakly* enforced. Given $\tilde{\mathbf{e}} \in \mathbf{E}^p_{\mathcal{T},\mathrm{nc}}$, keeping in mind that, along every edge E, the jump $[\Lambda_\mathcal{T}\tilde{\mathbf{e}}]_E$ is a polynomial of degree $\leq (p+1)$, we conclude that the chosen edge compatibility condition reads:

$$\begin{aligned} &\int_E [\Lambda_\mathcal{T}\tilde{\mathbf{e}}]_E\, q = 0 \quad \forall q \in \mathcal{P}_p(E),\ \forall E \in \mathcal{E} \quad \text{and} \\ &\int_E \Lambda_\mathcal{T}\tilde{\mathbf{e}}\, q = 0 \quad \forall q \in \mathcal{P}_p(E),\ \forall E \in \mathcal{E}_{\partial\Omega}. \end{aligned} \tag{3.4.9}$$

Remark 3.4.1. One could choose a priori the degree of the polynomials q between 0 and $p+1$. Indeed, a degree equal to $p+1$ defines conforming finite elements, because (3.4.9) then implies $[\Lambda_\mathcal{T}\tilde{\mathbf{e}}]_E = 0$ across all interior edges E, and $\Lambda_\mathcal{T}\tilde{e} = 0$ on $\partial\Omega$, and Lemma 3.3.1 leads to $\tilde{e} \in E^p_\mathcal{T}$. On the other hand, a degree strictly lower than $p+1$ in the implicit definition (3.4.9) of $E^p_{\mathcal{T},\mathrm{nc}}$ leads to a non-conforming finite element space, such that $E^p_\mathcal{T}$ is a strict subset of $E^p_{\mathcal{T},\mathrm{nc}}$. The degree p of the polynomials q, which is chosen here, yields an optimal order of convergence (see Theorem 3.4.5), whereas a degree strictly lower than p yields a sub-optimal order of convergence.

For any inner edge $E \in \mathcal{T}$, we may choose $q = 1$ in the left condition of (3.4.9) to obtain $\int_E [\Lambda_\mathcal{T}\tilde{\mathbf{e}}]_E = 0$. Let h_E denote the length of E. The combination of a Poincaré inequality with a trace inequality then yields

$$\begin{aligned} \|[\Lambda_\mathcal{T}\tilde{\mathbf{e}}]_E\|_{L^2(E)} &\leq Ch_E\, \|[\mathbf{t}_E \cdot \nabla_\mathcal{T}\Lambda_\mathcal{T}\tilde{\mathbf{e}}]_E\|_{L^2(E)} \\ &\overset{(3.4.7)}{=} Ch_E\, \|[\mathbf{t}_E \cdot \tilde{\mathbf{e}}]_E\|_{L^2(E)} \leq \tilde{C}h_E^{1/2}\, \|\tilde{\mathbf{e}}\|_{L^2(\omega_E)}. \end{aligned} \tag{3.4.10}$$

In a similar fashion we obtain for all boundary edges $E \in \mathcal{E}_{\partial\Omega}$ and all $\mathbf{e} \in \mathbf{E}^p_{\mathcal{T},\mathrm{nc}}$ the estimate

$$\|\Lambda_\mathcal{T}\tilde{\mathbf{e}}\|_{L^2(E)} \leq \tilde{C}h_E^{1/2}\, \|\tilde{\mathbf{e}}\|_{L^2(\omega_E)}. \tag{3.4.11}$$

These considerations are summarized in the following definition.

Definition 3.4.2. *Let the boundary of Ω be connected. The non-conforming intrinsic finite element space $\mathbf{E}^p_{\mathcal{T},\mathrm{nc}}$ is given by*

$$\mathbf{E}^p_{\mathcal{T},\mathrm{nc}} := \left\{ \mathbf{e} \in \mathbf{S}^{p,-1}_\mathcal{T} \mid \operatorname{curl}_\mathcal{T} \mathbf{e} = 0 \quad \textit{and} \quad \textit{(3.4.9) is satisfied} \right\}.$$

This definition directly implies that condition (3.4.2), i.e., $\mathbf{E}^p_{\mathcal{T}} \subset \mathbf{E}^p_{\mathcal{T},\mathrm{nc}}$ holds.
In Section 3.4.2 we will prove the following direct sum decomposition

$$\mathbf{E}^p_{\mathcal{T},\mathrm{nc}} = \mathbf{E}^p_{\mathcal{T}} \oplus \begin{cases} \bigoplus\limits_{E\in\mathcal{E}} \operatorname{span}\left\{\nabla_{\mathcal{T}} U^E_{p+1}\right\} & p \text{ even}, \\ \bigoplus\limits_{\tau\in\mathcal{T}} \operatorname{span}\left\{\nabla_{\mathcal{T}} U^\tau_{p+1}\right\} & p \text{ odd} \end{cases} \tag{3.4.12}$$

with functions U^E_{p+1} and U^τ_{p+1} defined in respectively (3.4.20) and (3.4.26). As a consequence, one deduces the following definition of the extended lifting operator.

Definition 3.4.3. *Let the boundary of Ω be connected. For a function $\mathbf{e} \in \mathbf{E}^p_{\mathcal{T},\mathrm{nc}}$ with*

$$\mathbf{e} = \mathbf{e}_1 + \begin{cases} \sum\limits_{E\in\mathcal{E}} \alpha_E \nabla_{\mathcal{T}} U^E_{p+1} & \text{if } p \text{ is even}, \\ \sum\limits_{\tau\in\mathcal{T}} \alpha_\tau \nabla_{\mathcal{T}} U^\tau_{p+1} & \text{if } p \text{ is odd} \end{cases} \tag{3.4.13}$$

for some $\mathbf{e}_1 \in \mathbf{E}^p_{\mathcal{T}}$ and real coefficients α_E resp. α_τ, the extended lifting operator $\Lambda_{\mathcal{T}}$ is given by

$$\Lambda_{\mathcal{T}}\mathbf{e} := \Lambda\mathbf{e}_1 + \begin{cases} \sum\limits_{E\in\mathcal{E}} \alpha_E U^E_{p+1} & \text{if } p \text{ is even}, \\ \sum\limits_{\tau\in\mathcal{T}} \alpha_\tau U^\tau_{p+1} & \text{if } p \text{ is odd}. \end{cases}$$

Proposition 3.4.4. *Let the boundary of Ω be connected. For any $\mathbf{e} \in \mathbf{E}^p_{\mathcal{T},\mathrm{nc}}$ with simply connected support $\omega_{\mathbf{e}} := \operatorname{supp}\mathbf{e}$, it holds*

$$\operatorname{supp}\Lambda_{\mathcal{T}}\mathbf{e} \subset \omega_{\mathbf{e}}.$$

Proof. We split $\mathbf{e} = \mathbf{e}_1 + \mathbf{e}_2$ according to (3.4.13) with $\mathbf{e}_1 \in \mathbf{E}$. Since the sum, in (3.4.12), is direct we conclude* that $\operatorname{supp}\mathbf{e}_i \subset \omega_{\mathbf{e}}$ for $i = 1, 2$. From Proposition 3.2.2 we obtain $\Lambda_{\mathcal{T}}\mathbf{e}_1 = \Lambda\mathbf{e}_1 \in H^1_0(\Omega)$. Since $\mathbf{e}_1|_{\Omega\backslash\omega_{\mathbf{e}}} = 0$ Poincaré's theorem implies that $\Lambda\mathbf{e}_1|_{\omega_i} = c_i$, i.e., is constant on each disjoint connected component ω_i of $\Omega\backslash\omega_{\mathbf{e}}$. Since $\omega_{\mathbf{e}}$ is simply connected, each component ω_i has an intersection $\overline{\omega_i} \cap \partial\Omega$ with positive length. The property $\Lambda\mathbf{e}_1 \in H^1_0(\Omega)$ implies that $\Lambda\mathbf{e}_1|_{\omega_i} = 0$. This proves $\operatorname{supp}\Lambda_{\mathcal{T}}\mathbf{e}_1 \subset \omega_{\mathbf{e}}$.
For even p, the definition of $\Lambda_{\mathcal{T}}$ for the non-conforming part $\mathbf{e}_2$ (in particular $\Lambda_{\mathcal{T}}\left(\nabla_{\mathcal{T}} U^E_{p+1}\right) = U^E_{p+1}$) implies that $\operatorname{supp}\nabla_{\mathcal{T}} U^E_{p+1} = \operatorname{supp} U^E_{p+1}$ so that $\operatorname{supp}\Lambda_{\mathcal{T}}\mathbf{e}_2 \subset \omega_{\mathbf{e}}$. The proof for odd p is by an analogous argument. □

Equipped with $\mathbf{E}^p_{\mathcal{T},\mathrm{nc}}$ and $\Lambda_{\mathcal{T}}$, the non-conforming Galerkin discretization of (3.2.6) reads: *Find $\mathbf{e}_{\mathcal{T}} \in \mathbf{E}^p_{\mathcal{T},\mathrm{nc}}$ such that*

$$\int_\Omega \varepsilon\mathbf{e}_{\mathcal{T}} \cdot \tilde{\mathbf{e}} = \int_\Omega \rho\Lambda_{\mathcal{T}}\tilde{\mathbf{e}} \qquad \forall\tilde{\mathbf{e}} \in \mathbf{E}^p_{\mathcal{T},\mathrm{nc}}. \tag{3.4.14}$$

We say that the exact solution $\mathbf{e} \in \mathbf{L}^2(\Omega)$ is piecewise smooth over a partition $\mathcal{P} = (\Omega_j)_{j=1}^J$ of Ω into J (possibly curved) polygons, if there exists some positive integer s such that

$$\mathbf{e}_{|\Omega_j} \in \mathbf{H}^s(\Omega_j) \quad \text{for } j = 1, 2, \ldots, J.$$

*Here, we also used the property that for a polynomial $q \in \mathbb{P}_p(\omega)$, $\omega \subset \Omega$ with positive area measure, it holds either $q|_\omega = 0$ or $\operatorname{supp} q = \omega$. In our application we choose $q = \mathbf{e}_1 + \mathbf{e}_2$ and apply the argument trianglewise.

We write $\mathbf{e} \in \mathbf{PH}^s(\Omega) = PH^s(\Omega) \times PH^s(\Omega)$ and refer for further properties and generalizations to non-integer values of s, e.g., to [51, Sec. 4.1.9].
For the approximation results, the finite element meshes $\mathcal{T}$ are assumed to be compatible with the partition $\mathcal{P}$ in the following sense: for all $\tau \in \mathcal{T}$, there exists a single index j such that $\mathring{\tau} \cap \Omega_j \neq \emptyset$.

Theorem 3.4.5. *Let the boundary of Ω be connected. Let the electrostatic permeability ε satisfy Assumption 3.2.3 and let $\rho \in L^2(\Omega)$. As an additional assumption on the regularity of the exact solution, we require that the exact solution of (3.2.6) satisfies $\mathbf{e} \in \mathbf{PH}^s(\Omega)$ for some positive integer s. Assume that the non-conforming finite element space $\mathbf{E}^p_{\mathcal{T},\mathrm{nc}}$ and the extended lifting operator $\Lambda_\mathcal{T}$ are defined on a compatible mesh $\mathcal{T}$, as in Definitions 3.4.2 and 3.4.3. Then, the non-conforming Galerkin discretization (3.4.14) has a unique solution which satisfies*

$$\|\mathbf{e} - \mathbf{e}_\mathcal{T}\|_{\mathbf{L}^2(\Omega)} \leq Ch^r \|\mathbf{e}\|_{PH^r(\Omega)}.$$

with $r := \min\{p+1, s\}$. The constant C only depends on $\varepsilon_{\min}$, $\varepsilon_{\max}$, $\|\varepsilon\|_{PW^{r,\infty}(\Omega)}$, p, and the shape regularity of the mesh.

Proof. The second Strang lemma applied to the non-conforming Galerkin discretization (3.4.14) implies the existence of a unique solution which satisfies the error estimate

$$\|\mathbf{e} - \mathbf{e}_\mathcal{T}\|_{\mathbf{L}^2(\Omega)} \leq \left(1 + \frac{\varepsilon_{\max}}{\varepsilon_{\min}}\right) \inf_{\tilde{\mathbf{e}} \in \mathbf{E}^p_{\mathcal{T},\mathrm{nc}}} \|\mathbf{e} - \tilde{\mathbf{e}}\|_{\mathbf{L}^2(\Omega)} + \frac{1}{\varepsilon_{\min}} \sup_{\tilde{\mathbf{e}} \in \mathbf{E}^p_{\mathcal{T},\mathrm{nc}} \setminus \{0\}} \frac{|\mathcal{L}_\mathbf{e}(\tilde{\mathbf{e}})|}{\|\tilde{\mathbf{e}}\|_{\mathbf{L}^2(\Omega)}},$$

where

$$\mathcal{L}_\mathbf{e}(\tilde{\mathbf{e}}) := \int_\Omega \varepsilon \mathbf{e} \cdot \tilde{\mathbf{e}} - \int_\Omega \rho \Lambda_\mathcal{T} \tilde{\mathbf{e}}.$$

The approximation properties of $\mathbf{E}^p_{\mathcal{T},\mathrm{nc}}$ are inherited from the approximation properties of $\mathbf{E}^p_\mathcal{T}$ in the first infimum because of the inclusion $\mathbf{E}^p_\mathcal{T} \subset \mathbf{E}^p_{\mathcal{T},\mathrm{nc}}$ in (3.4.2). For the second term we obtain

$$\mathcal{L}_\mathbf{e}(\tilde{\mathbf{e}}) = \int_\Omega \varepsilon (\nabla \Lambda \mathbf{e}) \cdot \tilde{\mathbf{e}} - \int_\Omega \rho \Lambda_\mathcal{T} \tilde{\mathbf{e}}. \tag{3.4.15}$$

Note that $\rho \in L^2(\Omega)$ implies that $\operatorname{div}(\varepsilon \nabla u) \in L^2(\Omega)$ and, in turn, that the jump $[\varepsilon \mathbf{e} \cdot \mathbf{n}_E]_E$ equals zero and the restriction $(\varepsilon \mathbf{e} \cdot \mathbf{n}_E)|_E$ is well defined. We may apply trianglewise integration by parts to (3.4.15) to obtain

$$\begin{aligned}
\mathcal{L}_\mathbf{e}(\tilde{\mathbf{e}}) &= \int_\Omega (\varepsilon \mathbf{e} \cdot \nabla_\mathcal{T} \Lambda_\mathcal{T} \tilde{\mathbf{e}} - \rho \Lambda_\mathcal{T} \tilde{\mathbf{e}}) \\
&= -\sum_{E \in \mathcal{E}} \int_E \varepsilon (\mathbf{e} \cdot \mathbf{n}_E) [\Lambda_\mathcal{T} \tilde{\mathbf{e}}]_E + \sum_{E \in \mathcal{E}_{\partial\Omega}} \int_E \varepsilon (\mathbf{e} \cdot \mathbf{n}_E) \Lambda_\mathcal{T} \tilde{\mathbf{e}}.
\end{aligned}$$

Let $q_E \in \mathcal{P}_p(E)$ denote the best approximation of $\varepsilon \mathbf{e} \cdot \mathbf{n}_E|_E$ with respect to the $L^2(E)$ norm. Then, the combination of (3.4.9) with standard approximation properties and a trace inequality leads to

$$
\begin{aligned}
\left|\mathcal{L}_{\mathbf{e}}\left(\tilde{\mathbf{e}}\right)\right| &= \left| -\sum_{E\in\mathcal{E}} \int_E \left(\varepsilon \frac{\partial u}{\partial \mathbf{n}_E} - q_E\right) \left[\Lambda_{\mathcal{T}}\tilde{\mathbf{e}}\right]_E + \sum_{E\in\mathcal{E}_{\partial\Omega}} \int_E \left(\varepsilon \frac{\partial u}{\partial \mathbf{n}_E} - q_E\right) \Lambda_{\mathcal{T}}\tilde{\mathbf{e}} \right| \\
&\leq \sum_{E\in\mathcal{E}} \left\| \varepsilon \frac{\partial u}{\partial \mathbf{n}_E} - q_E \right\|_{L^2(E)} \left\| \left[\Lambda_{\mathcal{T}}\tilde{\mathbf{e}}\right]_E \right\|_{L^2(E)} \\
&\qquad\qquad + \sum_{E\in\mathcal{E}_{\partial\Omega}} \left\| \varepsilon \frac{\partial u}{\partial \mathbf{n}_E} - q_E \right\|_{L^2(E)} \left\| \Lambda_{\mathcal{T}}\tilde{\mathbf{e}} \right\|_{L^2(E)} \\
&\leq C \left(\sum_{E\in\mathcal{E}} h_E^{r-1/2} \left\|\mathbf{e}\right\|_{\mathbf{H}^r(\tau_E)} \left\| \left[\Lambda_{\mathcal{T}}\tilde{\mathbf{e}}\right]_E \right\|_{L^2(E)} \right. \\
&\qquad\qquad \left. + \sum_{E\in\mathcal{E}_{\partial\Omega}} h_E^{r-1/2} \left\|\mathbf{e}\right\|_{\mathbf{H}^r(\tau_E)} \left\| \Lambda_{\mathcal{T}}\tilde{\mathbf{e}} \right\|_{L^2(E)} \right),
\end{aligned}
$$

where C depends only on p, $\|\varepsilon\|_{W^r(\tau_E)}$, and the shape regularity of the mesh, and τ_E is one triangle of ω_E. The estimates (3.4.10) - (3.4.11) along with the shape regularity of the mesh lead to the consistency estimate

$$
\begin{aligned}
\left|\mathcal{L}_{\mathbf{e}}\left(\tilde{\mathbf{e}}\right)\right| &\leq C \left(\sum_{E\in\mathcal{E}} h_E^{r} \left\|\mathbf{e}\right\|_{\mathbf{H}^r(\tau_E)} \left\|\tilde{\mathbf{e}}\right\|_{\mathbf{L}^2(\omega_E)} + \sum_{E\in\mathcal{E}_{\partial\Omega}} h_E^{r} \left\|\mathbf{e}\right\|_{\mathbf{H}^r(\tau_E)} \left\|\tilde{\mathbf{e}}\right\|_{\mathbf{L}^2(\omega_E)} \right) \\
&\leq \tilde{C} h^r \left\|\mathbf{e}\right\|_{\mathbf{PH}^r(\Omega)} \left\|\tilde{\mathbf{e}}\right\|_{\mathbf{L}^2(\Omega)},
\end{aligned}
$$

which completes the proof. □

Remark 3.4.6. If one chooses in (3.4.9) a degree $p' < p$ for the test-polynomials q, then the order of convergence behaves like $h^{r'} \|\mathbf{e}\|_{\mathbf{H}^{r'}(\Omega)}$, with $r' := \min\{p'+1, s\}$, because the best approximation q_E now belongs to $\mathcal{P}_{p'}(E)$.

3.4.2 A Local Basis for Non-Conforming Intrinsic Finite Elements

Like in Proposition 3.3.4, we construct the space $\mathbf{E}^p_{\mathcal{T},\mathrm{nc}}$ by defining basis functions whose supports are given by a single triangle $\tau \in \mathcal{T}$, edge-oriented basis functions whose supports are given by ω_E, for $E \in \mathcal{E}$, and vertex-oriented basis functions whose supports are given by ω_V, $V \in \mathcal{V}$. The definitions of ω_E and ω_V are given in (2.2.11) and (2.2.12).
The corresponding spaces spanned by these basis functions are denoted by $\mathbf{B}^p_{\tau,\mathrm{nc}}$, $\mathbf{B}^p_{E,\mathrm{nc}}$ and $\mathbf{B}^p_{V,\mathrm{nc}}$.
In Theorem 3.4.13, we will prove that $\mathbf{E}^p_{\mathcal{T},\mathrm{nc}}$ can be decomposed into a direct sum of these local subspaces.

Triangle Supported Basis Functions

In this section, let $\tau \in \mathcal{T}$ denote any fixed triangle in the mesh. The Lagrange basis of $\mathcal{P}_p(\tau)$ with respect to $\mathcal{N}^p \cap \tau$ is denoted by $b^\tau_{p,N}$, $N \in \mathcal{N}^p \cap \tau$, and is characterized by

$$
b^\tau_{N,p} \in \mathcal{P}_p(\tau) \quad \text{and} \quad \forall N' \in \mathcal{N}^p \cap \tau \quad b^\tau_{N,p}\left(N'\right) = \begin{cases} 1 & \text{if } N = N', \\ 0 & \text{if } N \neq N'. \end{cases} \tag{3.4.16}
$$

We denote the (discontinuous in general) extension by zero of $b^{\tau}_{p,N}$ to $\Omega\backslash\tau$ again by $b^{\tau}_{p,N}$. From Lemma 3.3.2 and Conditions (3.4.2), (3.4.9), we deduce

$$\mathbf{B}^p_{\tau,\mathrm{nc}} = \Big\{ \mathbf{e}|_\tau \in \nabla\mathcal{P}_{p+1}(\tau) \mid \operatorname{supp}\mathbf{e} \subset \tau \quad \text{and} \quad \forall E \subset \partial\tau,\ \forall q \in \mathcal{P}_p(E) : \int_E q\Lambda_{\mathcal{T}}\mathbf{e} = 0 \Big\}. \tag{3.4.17}$$

According to (3.4.17), it is clear that $\mathbf{B}^p_\tau \subset \mathbf{B}^p_{\tau,\mathrm{nc}}$. In the next step, we use the compatibility conditions in (3.4.17) for the explicit characterization of $\mathbf{B}^p_{\tau,\mathrm{nc}}$.

Lemma 3.4.7. *Let the boundary of Ω be connected. For $\tau \in \mathcal{T}$, the non-conforming finite element space $\mathbf{B}^p_{\tau,\mathrm{nc}}$ is given by*

$$\mathbf{B}^p_{\tau,\mathrm{nc}} = \begin{cases} \mathbf{B}^p_\tau & \text{if } p \text{ is even,} \\ \mathbf{B}^p_\tau + \operatorname{span}\left\{\nabla_{\mathcal{T}} U^\tau_{p+1}\right\} & \text{if } p \text{ is odd,} \end{cases} \tag{3.4.18}$$

where U^τ_{p+1} is defined in (3.4.20).

Proof. Pick some $\mathbf{e} \in \mathbf{B}^p_{\tau,\mathrm{nc}}$, let $u := \Lambda_{\mathcal{T}}\mathbf{e}$ and denote the restrictions to τ by $\mathbf{e}_\tau$ and u_τ. For $E \in \mathcal{E} \cup \mathcal{E}_{\partial\Omega}$, let χ_E be as in (3.4.8) the affine pullback to $[-1,1]$. Let $L_p : [-1,1] \to \mathbb{R}$ denote the Legendre polynomials of degree p with the normalization convention that $L_p(1) = 1$. In turn, this implies $L_p(-1) = (-1)^p$. We lift them to the edge E via $L^E_p := L_p \circ \chi_E^{-1}$. It is well known that L^E_{p+1} satisfies the orthogonality condition

$$(L^E_{p+1}, q)_{L^2(E)} = 0 \qquad \forall q \in \mathcal{P}_p(E).$$

The compatibility condition in (3.4.17) therefore implies, for all $E \subset \partial\tau$, that

$$u_\tau|_E = c_E \cdot L^E_{p+1} \quad \text{for some } c_E \in \mathbb{R}. \tag{3.4.19}$$

The relation $u_\tau \in \mathcal{P}_{p+1}(\tau)$ implies that $u_\tau|_{\partial\tau}$ is continuous so that u_τ is continuous at every vertex of τ. We distinguish two cases.

Let p be even. In this case we have $L_{p+1}(1) = -L_{p+1}(-1)$ so that the continuity at the vertices of τ implies $c_E = 0$. Thus $u_\tau|_{\partial\tau} = 0$ and we have proved (3.4.18) for even p.

Let p be odd. Now we have $L_{p+1}(1) = L_{p+1}(-1)$ so that $c_E = c_\tau$ for all $E \subset \partial\tau$ and some fixed c_τ. For any $N \in \mathcal{N}^{p+1} \cap \partial\tau$, we denote by $E_N \subset \partial\tau$ a fixed, but arbitrary, edge such that $N \in E_N$. We define the function (cf. Figure 3.2)

$$U^\tau_{p+1} := \sum_{N \in \mathcal{N}^{p+1} \cap \partial\tau} L^{E_N}_{p+1}(N)\, b^\tau_{p+1,N} \tag{3.4.20}$$

whose gradient $\nabla_{\mathcal{T}} U^\tau_{p+1}$ satisfies the compatibility condition across the edges. This leads to the assertion for odd p.

□

Remark 3.4.8. The space $\mathbf{B}^p_{\tau,\mathrm{nc}}$ satisfies the compatibility conditions (3.4.9). A basis of $\mathbf{B}^p_{\tau,\mathrm{nc}}$ for even p is given by $\left\{\nabla_{\mathcal{T}} b^{\mathcal{T}}_{p+1,N} : N \in \mathcal{N}^{p+1} \cap \overset{\circ}{\tau}\right\}$, while a basis for odd p is given by $\left\{\nabla_{\mathcal{T}} b^{\mathcal{T}}_{p+1,N} : N \in \mathcal{N}^{p+1} \cap \overset{\circ}{\tau}\right\} \cup \left\{\nabla_{\mathcal{T}} U^\tau_{p+1}\right\}$.

Taking into account the Remark 3.4.8, the triangle-oriented subspace $\mathbf{B}^p_{\tau,\mathrm{nc}}$ is defined by:

$$\mathbf{B}^p_{\tau,\mathrm{nc}} = \begin{cases} span\{\nabla_{\mathcal{T}} b^{\mathcal{T}}_{p+1,N} : N \in \mathcal{N}^{p+1} \cap \overset{\circ}{\tau}\} & \text{if } p \text{ is even,} \\ span\left\{\{\nabla_{\mathcal{T}} b^{\mathcal{T}}_{p+1,N} : N \in \mathcal{N}^{p+1} \cap \overset{\circ}{\tau}\} \cup \{\nabla_{\mathcal{T}} U^\tau_{p+1}\}\right\} & \text{if } p \text{ is odd,} \end{cases} \tag{3.4.21}$$

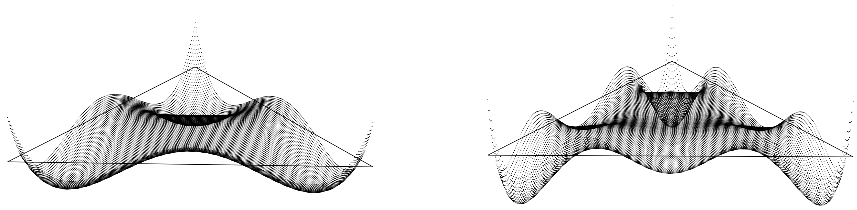

Figure 3.2: *Representation of U^{τ}_{p+1} for $p = 3$ (left) and $p = 5$ (right)*

Edge-oriented Basis Functions

Lemma 3.4.9. *Let the boundary of Ω be connected. For $E \in \mathcal{E}$, the non-conforming finite element space $\mathbf{B}^p_{E,\mathrm{nc}}$ is explicitly given by*

$$\mathbf{B}^p_{E,\mathrm{nc}} = \begin{cases} \mathbf{B}^p_E + \operatorname{span}\left\{\nabla_{\mathcal{T}} U^E_{p+1}\right\} & \text{if } p \text{ is even,} \\ \mathbf{B}^p_E & \text{if } p \text{ is odd,} \end{cases} \tag{3.4.22}$$

where U^E_{p+1} is defined in (3.4.26).

Proof. Given $\mathbf{e} \in \mathbf{B}^p_E$, it follows from (3.3.14) that $\operatorname{supp} \mathbf{e} \subset \omega_E$, without being restricted to a single triangle (otherwise, $\mathbf{e} \in \mathbf{B}^p_\tau$ for some $\tau \in \mathcal{T}_E$). Taking into account that the spaces $\mathbf{B}^p_{\tau,\mathrm{nc}}$, $\mathbf{B}^p_{E,\mathrm{nc}}$ are spanned by triangle- and edge- oriented basis functions it follows that $\mathbf{e} \in \mathbf{B}^p_{E,\mathrm{nc}}$. Hence, $\mathbf{B}^p_E \subset \mathbf{B}^p_{E,\mathrm{nc}}$.
Let $E \in \mathcal{E}$ be an arbitrary edge. Since any $\mathbf{e} \in \mathbf{B}^p_{E,\mathrm{nc}}$ can be expressed locally on $\tau \in \mathcal{T}_E$ by $\mathbf{e}|_\tau = \nabla v_\tau$ for some $v_\tau \in \mathcal{P}_{p+1}(\tau)$ (cf. Lemma 3.3.2)) we have

$$\mathbf{B}^p_{E,\mathrm{nc}} \subset \bigoplus_{\tau \in \mathcal{T}_E} \operatorname{span}\left\{\nabla_{\mathcal{T}} b^\tau_{N,p+1} \mid N \in \mathcal{N}^{p+1} \cap \tau\right\},$$

where we recall that $b^\tau_{N,p+1}$ are the Lagrange basis functions on τ and vanish on $\Omega \backslash \tau$. Since the functions $b^\tau_{N,p+1}$ for the inner nodes $N \in \mathcal{N}^{p+1} \cap \mathring{\tau}$ belong to the space $\mathbf{B}^p_{\tau,\mathrm{nc}}$, we obtain

$$\mathbf{B}^p_{E,\mathrm{nc}} \subset \bigoplus_{\tau \in \mathcal{T}_E} \operatorname{span}\left\{\nabla_{\mathcal{T}} b^\tau_{N,p+1} \mid N \in \mathcal{N}^{p+1} \cap \partial\tau\right\}.$$

For $\mathbf{e} \in \mathbf{B}^p_{E,\mathrm{nc}}$, let $u := \Lambda_{\mathcal{T}} \mathbf{e}$ and $u_\tau := u|_\tau$, $\tau \in \mathcal{T}_E$. By arguing as in the case of triangle-supported basis functions, we derive from the compatibility conditions (3.4.9)

$$[u]_E = c_E L^E_{p+1} \quad \text{and} \quad \forall E' \subset \partial\omega_E \quad u|_{E'} = c_{E'} L^{E'}_{p+1}. \tag{3.4.23}$$

Again, the relation $u_\tau \in \mathcal{P}_{p+1}(\tau)$ implies the continuity of u_τ at the vertices of τ.
Let p be even. The continuity of u_τ along $\partial\tau$ and the endpoint properties of $L^{E'}_{p+1}$ imply that $u_\tau\left(A^E\right) = u_\tau\left(B^E\right)$ for $\tau \in \mathcal{T}_E$, where A^E, B^E denote the endpoints of E (cf. Figure 3.3).

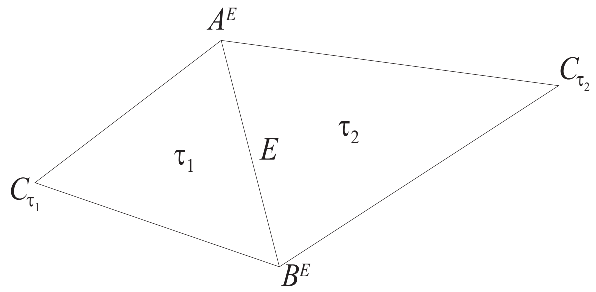

Figure 3.3: *Edge $E \in \mathcal{E}$ with endpoints A^E, B^E and two neighboring triangles τ_1, τ_2,*

Hence, $[u]_E\left(A^E\right) = [u]_E\left(B^E\right)$. Since $L^E_{p+1}\left(A^E\right) = -L^E_{p+1}\left(B^E\right)$ we conclude from the first condition in (3.4.23) that $c_E = 0$ holds so that u is continuous across E. Recall that the edges are closed and define

$$b^E_{p+1,N} := \begin{cases} \left. b^{\mathcal{T}}_{p+1,N}\right|_{\omega_E} & \text{on } \omega_E, \\ 0 & \text{on } \Omega\backslash\omega_E, \end{cases} \tag{3.4.24}$$

where $b^{\mathcal{T}}_{p+1,N}$ are as in (3.3.8). The space $\mathbf{R}^p_{E,\text{nc}}$ is constructed such that the following direct sum decomposition holds:

$$\mathbf{B}^p_{E,\text{nc}} = \mathbf{B}^p_E \oplus \mathbf{R}^p_{E,\text{nc}}. \tag{3.4.25}$$

Note that then

$$\mathbf{R}^p_{E,\text{nc}} \subset \operatorname{span}\left\{\nabla_{\mathcal{T}} b^E_{p+1,N} \mid N \in \mathcal{N}^{p+1} \cap \partial\omega_E\right\}.$$

Pick $\mathbf{e} \in \mathbf{R}^p_{E,\text{nc}}$ and set $u := \Lambda_{\mathcal{T}}\mathbf{e}$. The continuity property $[u]_E = 0$ which we already derived implies that $c_{E'} = c$ for all $E' \subset \partial\omega_E$. This leads to $u = cU^E_{p+1}$ with U^E_{p+1} designed as (cf. Figure 3.4)

$$U^E_{p+1} := \sum_{N \in \mathcal{N}^{p+1} \cap \partial\omega_E} L^{E_N}_{p+1}(N)\, b^E_{p+1,N} \quad \text{and} \quad b^E_{p+1,N} \text{ as in (3.4.24)}, \tag{3.4.26}$$

where, again, for $N \in \mathcal{N}^{p+1} \cap \partial\omega_E$ we assign some edge $E_N \subset \partial\omega_E$ such that $N \in E_N$. Hence $\mathbf{R}^p_{E,\text{nc}} = \operatorname{span}\left\{\nabla_{\mathcal{T}} U^E_{p+1}\right\}$ and the assertion follows for even p.
Let p be odd. We have

$$\mathbf{B}^p_{E,\text{nc}} = \mathbf{B}^p_E \oplus \mathbf{R}^p_{E,\text{nc}}, \tag{3.4.27}$$

Pick $\mathbf{e} \in \mathbf{R}^p_{E,\text{nc}}$ and set $u := \Lambda_{\mathcal{T}}\mathbf{e}$. For any edge $E' \subset \partial\omega_E \cap \partial\tau$, the restriction of u_τ to E' must be a multiple of a Legendre polynomial. The continuity of u_τ along $\partial\tau$ implies in particular the continuity at C_{τ_i}, $i \in \{1,2\}$ (cf. Figure 3.3). Hence, $\left. u_\tau\right|_{\partial\omega_E \cap \partial\tau} = c_\tau \left. U^\tau_{p+1}\right|_{\partial\omega_E \cap \partial\tau}$ for some c_τ and U^τ_{p+1} as defined in (3.4.20), and

$$\tilde{u} = u - \sum_{\tau \in \mathcal{T}_E} c_\tau U^\tau_{p+1}$$

vanishes at $\partial\omega_E$. Since the jump of $\tilde{u}$ across E vanishes in A^E and B^E the first condition in (3.4.23) implies that $\tilde{u}$ is continuous in ω_E and vanishes on $\partial\omega_E$. From this we conclude that $\tilde{u} \in \mathbf{B}^p_E$. The characterization of $\mathbf{R}^p_{E,\text{nc}}$ as a direct sum in (3.4.27) shows that $u = 0$ and thus $\mathbf{R}^p_{E,\text{nc}} = \{0\}$. $\square$

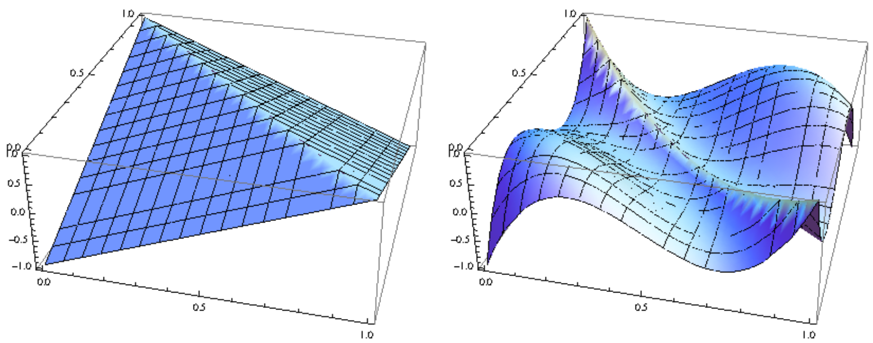

Figure 3.4: *Representation of U_{p+1}^{E} for $p=0$ (left) and $p=2$ (right)*

Remark 3.4.10. The space $\mathbf{B}_{E,\mathrm{nc}}^{p}$ satisfies the compatibility conditions (3.4.9). A basis of $\mathbf{B}_{E,\mathrm{nc}}^{p}$ for odd p is given by $\left\{\nabla_{\mathcal{T}} b_{p+1,N}^{\mathcal{T}} : N \in \mathcal{N}^{p+1} \cap \overset{\circ}{E}\right\}$ while for even p we may choose $\left\{\nabla_{\mathcal{T}} b_{p+1,N}^{\mathcal{T}} : N \in \mathcal{N}^{p+1} \cap \overset{\circ}{E}\right\} \cup \left\{\nabla_{\mathcal{T}} U_{p+1}^{E}\right\}$.

The Remark 3.4.10 allows us to define the edge-oriented subspace $\mathbf{B}_{E,\mathrm{nc}}^{p}$ as:

$$\mathbf{B}_{E,\mathrm{nc}}^{p} = \begin{cases} span\left\{\{\nabla_{\mathcal{T}} b_{p+1,N}^{\mathcal{T}} : N \in \mathcal{N}^{p+1} \cap \overset{\circ}{E}\} \cup \{\nabla_{\mathcal{T}} U_{p+1}^{E}\}\right\} & \text{if } p \text{ is even,} \\ span\{\nabla_{\mathcal{T}} b_{p+1,N}^{\mathcal{T}} : N \in \mathcal{N}^{p+1} \cap \overset{\circ}{E}\} & \text{if } p \text{ is odd.} \end{cases} \tag{3.4.28}$$

Vertex-oriented Basis Functions

In this section we will find an explicit representation of the vertex-oriented subspace $\mathbf{B}_{V,\mathrm{nc}}^{p}$.

Lemma 3.4.11. *Let the boundary of Ω be connected. It holds*

$$\mathbf{B}_{V,\mathrm{nc}}^{p} = \begin{cases} \{0\} & \text{if } p \text{ is even,} \\ \mathbf{B}_{V}^{p} & \text{if } p \text{ is odd.} \end{cases} \tag{3.4.29}$$

Proof. In a first step, we will prove that the subspace $\mathbf{R}_{p+1,V}^{\mathcal{T}}$, which is constructed accordingly to the next direct sum decomposition

$$\mathbf{R}_{p+1,V}^{\mathcal{T}} \oplus \bigoplus_{E\in\mathcal{E}_V} \mathbf{B}_{E,\mathrm{nc}}^{p} \oplus \bigoplus_{\tau\in\mathcal{T}_V} \mathbf{B}_{\tau,\mathrm{nc}}^{p} = \left\{\mathbf{e}' \in \mathbf{E}_{\mathcal{T},\mathrm{nc}}^{p} \middle| \operatorname{supp}\mathbf{e}' \subset \omega_V\right\}, \tag{3.4.30}$$

satisfies

$$\mathbf{R}_{p+1,V}^{\mathcal{T}} \subset \mathbf{B}_{V}^{p}. \tag{3.4.31}$$

In the second step, we will show that for even p the inclusion

$$\mathbf{B}_{V}^{p} \subset \bigoplus_{E\in\mathcal{E}_V} \mathbf{B}_{E,\mathrm{nc}}^{p} \oplus \bigoplus_{\tau\in\mathcal{T}_V} \mathbf{B}_{\tau,\mathrm{nc}}^{p} \tag{3.4.32}$$

holds so that the first case in (3.4.29) follows. In the case of odd p we first note that $\mathbf{B}_{V}^{p} = \operatorname{span}\left\{\nabla b_{p+1,V}^{\mathcal{T}}\right\}$. We will prove that, for all $V \in \mathcal{V}$ (cf. (3.4.22)),

$$\nabla b_{p+1,V}^{\mathcal{T}} \notin \bigoplus_{E\in\mathcal{E}_V} \mathbf{B}_{E,\mathrm{nc}}^{p} \oplus \bigoplus_{\tau\in\mathcal{T}_V} \mathbf{B}_{\tau,\mathrm{nc}}^{p}. \tag{3.4.33}$$

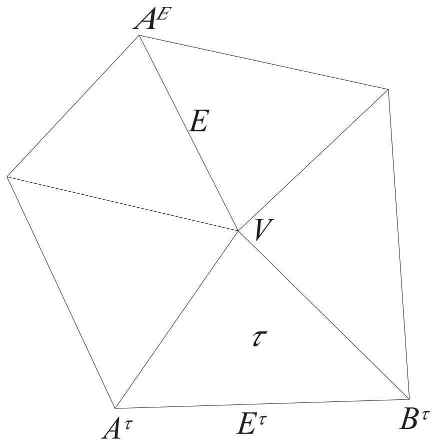

Figure 3.5: *A vertex $V \in \mathcal{V}$, neighboring triangle $\tau \in \mathcal{T}_V$, and neighboring edge $E \in \mathcal{T}_V$.*

From (3.4.31), we conclude that $\mathbf{R}^{\mathcal{T}}_{p+1,V} = \mathbf{B}^p_V$.

1st Step. Choose any

$$\mathbf{e} \in \left\{ \mathbf{e}' \in \mathbf{E}^p_{\mathcal{T},\mathrm{nc}} | \operatorname{supp} \mathbf{e}' \subset \omega_V \right\} \tag{3.4.34}$$

and set $u := \Lambda_{\mathcal{T}} \mathbf{e}$.

Let p be odd. For $\tau \in \mathcal{T}_V$, the edge E^τ is given by the condition $E^\tau \subset \partial\tau \cap \partial\omega_V$ (cf. Figure 3.5).

Since $L^{E^\tau}_{p+1}$ has even degree the values at the endpoints A^τ, B^τ of E^τ equal one. We set $u_\tau := u|_\tau$ and define

$$\tilde{u} := u - \sum_{\tau \in \mathcal{T}_V} u_\tau \left(A^\tau \right) U^\tau_{p+1}.$$

Hence, $\tilde{u} = 0$ on $\partial\omega_V$. Any edge $E \in \mathcal{E}_V$ has V as one endpoint; denote the other one by A^E. We employ the condition $[\tilde{u}]_E = c_E L^E_{p+1}$ at the point A^E to obtain $c_E = 0$. Hence $\tilde{u}$ is continuous and vanishes on $\partial\omega_V$. Consequently, $\tilde{u}$ is a conforming function, i.e.,

$$\begin{aligned}\nabla \left(u - \sum_{\tau \in \mathcal{T}_V} u_\tau \left(A^\tau \right) U^\tau_{p+1} \right) &\in \mathbf{B}^p_V \oplus \bigoplus_{E \in \mathcal{E}_V} \mathbf{B}^p_E \oplus \bigoplus_{\tau \in \mathcal{T}_V} \mathbf{B}^p_\tau \\ &\subset \mathbf{B}^p_V \oplus \bigoplus_{E \in \mathcal{E}_V} \mathbf{B}^p_{E,\mathrm{nc}} \oplus \bigoplus_{\tau \in \mathcal{T}_V} \mathbf{B}^p_{\tau,\mathrm{nc}}.\end{aligned}$$

Hence, (3.4.30) implies $\mathbf{R}^{\mathcal{T}}_{p+1,V} \subset \mathbf{B}^p_V$.

Let p be even. We number the edges in $\mathcal{E}_V$ counter-clockwise. $\mathcal{E}_V = \{E_1, \ldots, E_q\}$ (see Figure 3.6) for some q and, to simplify the notation, we set $E_0 := E_q$ and $E_{q+1} := E_1$.

The triangle which has E_{i-1} and E_i as edges and V as a vertex is denoted by τ_i. Each edge E_i has V as an endpoint; denote by A_i the other one. We further set $E^{\mathrm{out}}_i := \partial\tau_i \cap \partial\omega_V$. We define recursively $u_0 := u$ and, for $k = 1, 2, \ldots, q$,

$$u_k = u_{k-1} - \frac{\left(u_{k-1}\right)_{\tau_k} \left(A_k\right)}{U^{E_k}_{p+1} \left(A_k\right)} U^{E_k}_{p+1}.$$

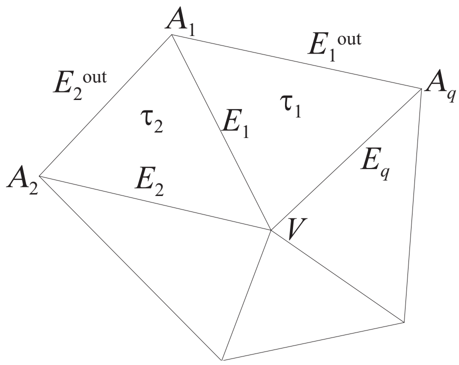

Figure 3.6: *Vertex $V \in \mathcal{V}$ and outgoing edges – numbered counterclockwise. The triangles $\tau_i \in \mathcal{T}_V$ contain E_{i-1}, E_i as edges and V as a vertex.*

Note that $u_q = 0$ on $\partial\omega_V \backslash E_1^{\text{out}}$. By arguing as for the case of odd p we deduce that u_q is continuous on $\omega_V \backslash E_1$. Since $u_q|_{E_1^{\text{out}}} = c_1 L_{p+1}^{E_1^{\text{out}}}$ for some $c_1 \in \mathbb{R}$, the property $u_q(A_q) = 0$ and $L_{p+1}^{E_1^{\text{out}}}(A_q) \neq 0$ implies $c_1 = 0$. Hence, $u_q|_{\partial\omega_V} = 0$. Arguing as in the case of odd p finally yields that u_q is continuous on ω_V and the assertion follows as in the case of odd p. This finishes the proof of (3.4.31).

2nd Step: To prove (3.4.32) we again distinguish between even and odd values of p.
Let p be even. In this section, let $b_{p,N}^{\tau}$ be the Lagrange basis defined in (3.4.16). We introduce the function $b_{p+1,N}^{\omega_V}$ defined as

$$b_{p+1,N}^{\omega_V} := \begin{cases} b_{p+1,N}^{\tau}, \ \forall \tau \in \omega_V \\ 0, \ \text{in } \Omega \setminus \omega_V \end{cases} \tag{3.4.35}$$

The function $b_{p+1,N}^{\omega_V}$ is continuous on ω_V.
Then, by using U_{p+1}^{E} as in (3.4.26), we define a function

$$w_1 := b_{p+1,V}^{\omega_V} - \frac{1}{q}\sum_{E\in\mathcal{E}_V} U_{p+1}^{E}(V)\,U_{p+1}^{E} - \frac{1}{q}\sum_{N\in\partial\omega_V\cap\mathcal{N}^{p+1}} b_{p+1,N}^{\omega_V} \tag{3.4.36}$$

which is continuous in ω_V and vanishes at V and at all inner nodes $\mathcal{N}^{p+1} \cap \mathring{\tau}$, $\tau \in \mathcal{T}_V$.
In any vertex A_i of ω_V only three terms in the first sum from the definition of w_1 are different of zero, more precisely:

$$\sum_{E\in\mathcal{E}_V} U_{p+1}^{E}(V)\,U_{p+1}^{E}(A_i) = U_{p+1}^{E_{i-1}}(V)U_{p+1}^{E_{i-1}}(A_i) + U_{p+1}^{E_i}(V)U_{p+1}^{E_i}(A_i) + U_{p+1}^{E_{i+1}}(V)U_{p+1}^{E_{i+1}}(A_i) = -1$$

Therefore $w_1(A_i) = 0$ for all $i \in \{1,\dots,q\}$ and we conclude that $w_1 = 0$ on $\partial\omega_V$.
Next, the function

$$w_2 := w_1 - \sum_{E\in\mathcal{E}_V}\ \sum_{N\in\mathcal{N}^{p+1}\cap\mathring{E}} w_1(N)\, b_{p+1,N}^{\mathcal{T}} \tag{3.4.37}$$

vanishes at all nodal points $\mathcal{N}^{p+1} \cap \left(\mathring{\omega}_E\right)$ and the jumps across $E \in \mathcal{E}_V$ have to vanish due to the compatibility condition. Since w_1 as well as the basis functions in the sum (3.4.37) vanish along $\partial\omega_E$, we conclude that w_2 vanishes also on $\partial\omega_E$ and thus $w_2 = 0$ in Ω. Hence, we have established (3.4.32), or, more precisely, that

$$\nabla b_{p+1,V}^{\mathcal{T}} \in \bigoplus_{E\in\mathcal{E}_V} \mathbf{B}_{E,\mathrm{nc}}^{p}.$$

Let p be odd. We will prove (3.4.33) by contradiction and assume that

$$\nabla b_{p+1,V}^{\mathcal{T}} \in \bigoplus_{E\in\mathcal{E}_V} \mathbf{B}_{E,\mathrm{nc}}^{p} \oplus \bigoplus_{\tau\in\mathcal{T}_V} \mathbf{B}_{\tau,\mathrm{nc}}^{p}.$$

We then infer from Remark 3.4.8 and Remark 3.4.10 that

$$b_{p+1,V}^{\mathcal{T}} = \underbrace{\sum_{N\in\mathcal{N}^{p+1}\backslash\mathcal{V}} \alpha_N b_{p+1,N}^{\mathcal{T}}}_{=:v_{\mathrm{c}}} + \underbrace{\sum_{\tau\in\mathcal{T}} \alpha_\tau U_{p+1}^{\tau}}_{v_{\mathrm{nc}}} \tag{3.4.38}$$

for some real coefficients α_N and α_τ. Since $b_{p+1,N}^{\mathcal{T}}$ and v_{c} are continuous in Ω, the function v_{nc} must also be continuous. By contradiction it is easy to prove that

$$C^0(\Omega) \cap \bigoplus_{\tau\in\mathcal{T}} \operatorname{span}\left\{U_{p+1}^{\tau}\right\} = \operatorname{span}\left\{U_{p+1}\right\} \text{ with } U_{p+1} := \sum_{\tau\in\mathcal{T}} U_{p+1}^{\tau},$$

so that $v_{\mathrm{nc}} \in \operatorname{span}\{U_{p+1}\}$. Since $v_{\mathrm{c}}(V) = 0$ and $b_{p+1,V}^{\mathcal{T}}(V) = 1$, we obtain from (3.4.38) that $v_{\mathrm{nc}}(V) = 1$. The restriction of U_{p+1} to any edge $E \in \mathcal{E} \cup \mathcal{E}_{\partial\Omega}$ is a Legendre polynomial of even degree, which implies $v_{\mathrm{nc}}(V') = 1$, for every $V' \in \mathcal{V} \cup \mathcal{V}_{\partial\Omega}$. But the functions $b_{p+1,V}^{\mathcal{T}}$ and v_{c} vanish on $\partial\Omega$. This contradicts $v_{\mathrm{nc}}(V') = 1$ for the boundary points $V' \in \mathcal{V}_{\partial\Omega}$. □

Remark 3.4.12. The space $\mathbf{B}_{V,\mathrm{nc}}^{p}$ satisfies the compatibility conditions (3.4.9). A basis of $\mathbf{B}_{V,\mathrm{nc}}^{p}$ for odd p is given by $\left\{\nabla_{\mathcal{T}} b_{p+1,V}^{\mathcal{T}} : V \in \mathcal{V}\right\}$ while for even p we have $\mathbf{B}_{V,\mathrm{nc}}^{p} = \{0\}$.

Therefore, the definition of the vertex-oriented subspace $\mathbf{B}_{V,\mathrm{nc}}^{p}$ is given by:

$$\mathbf{B}_{V,\mathrm{nc}}^{p} = \begin{cases} \{0\} & \text{if } p \text{ is even,} \\ \mathit{span}\{\nabla_{\mathcal{T}} b_{p+1,V}^{\mathcal{T}} : V \in \mathcal{V}\} & \text{if } p \text{ is odd.} \end{cases} \tag{3.4.39}$$

Properties of the Non-Conforming Intrinsic Basis functions

Theorem 3.4.13. *Let the boundary of Ω be connected. A basis of $\mathbf{E}_{\mathcal{T},\mathrm{nc}}^{p}$ is given by*

$$\left\{\nabla_{\mathcal{T}} b_{p+1,N}^{\mathcal{T}} : N \in \mathcal{N}^{p+1}\backslash\mathcal{V}\right\} \cup \bigcup_{E\in\mathcal{T}} \left\{\nabla_{\mathcal{T}} U_{p+1}^{E}\right\} \quad \textit{if } p \textit{ is even,} \tag{3.4.40}$$

and by

$$\left\{\nabla_{\mathcal{T}} b_{p+1,N}^{\mathcal{T}} : N \in \mathcal{N}^{p+1}\right\} \cup \bigcup_{\tau\in\mathcal{T}} \left\{\nabla_{\mathcal{T}} U_{p+1}^{\tau}\right\} \quad \textit{if } p \textit{ is odd.} \tag{3.4.41}$$

Remark 3.4.14. At first glance, it seems that $B_V^p \not\subset E_{\mathcal{T},\mathrm{nc}}^p$ for even p. Actually, this subspace of $E_{\mathcal{T}}^p$ has already been taken into account; see (3.4.32).

Proof of Theorem 3.4.13. By construction, the space $\widetilde{\mathbf{E}^p_{\mathcal{T},\mathrm{nc}}}$ of the functions found in (3.4.40) as in (3.4.41) is a subspace of $\mathbf{E}^p_{\mathcal{T},\mathrm{nc}}$. So, it remains to prove $\mathbf{E}^p_{\mathcal{T},\mathrm{nc}} \subset \widetilde{\mathbf{E}^p_{\mathcal{T},\mathrm{nc}}}$.
Let p be odd. The arguments in the following are very similar to those in the proof of Lemma 3.4.11 for odd p. Let $u := \Lambda_{\mathcal{T}}\mathbf{e}$. Pick some $\tau \in \mathcal{T}$ having at least one edge on $\partial\Omega$. Condition (3.4.9) implies that for all edges $E \subset \partial\tau \cap \partial\Omega$, the restriction $u|_E$ is a multiple of the Legendre polynomial L^E_{p+1}. The continuity of $u|_\tau$ on τ implies that there exists a function $\tilde{u} := cU^\tau_{p+1}$ with $\nabla\tilde{u} \in \mathbf{B}^p_{\tau,\mathrm{nc}}$ for some c such that $u_1 := u - \tilde{u}$ satisfies $u_1|_{\partial\tau\cap\partial\Omega} = 0$. Since u_1 vanishes at the endpoints of all such edges $E \in \mathcal{E}_{\partial\Omega}$, the function u_1 is also continuous across the other edges $E \subset \partial\tau \cap \Omega$. Let

$$\tilde{u}_1 = \sum_{N \in \mathcal{N}^{p+1} \cap \overset{\circ}{\tau}} u_1(N)\, b^{\mathcal{T}}_{p+1,N} + \sum_{E \subset \partial\tau\cap\Omega} \sum_{N \in \mathcal{N}^{p+1} \cap \overset{\circ}{E}} u_1(N)\, b^{\mathcal{T}}_{p+1,N} + \sum_{V \in \partial\tau\cap\Omega} u_1(V)\, b^{\mathcal{T}}_{p+1,V}$$

and note that $\tilde{u}_1 \in \widetilde{\mathbf{E}^p_{\mathcal{T},\mathrm{nc}}}$. In particular Lemma 3.4.11 implies that $b^{\mathcal{T}}_{p+1,V} \in \widetilde{\mathbf{E}^p_{\mathcal{T},\mathrm{nc}}}$. Note that $u_2 := u_1 - \tilde{u}_1$ vanishes on τ. Iterating this construction for the remaining triangles finally results in a function that vanishes on Ω. Thus we have found a linear representation of u by functions in $\widetilde{\mathbf{E}^p_{\mathcal{T},\mathrm{nc}}}$.
Let p be even. Again the arguments are very similar to those in the proof of Lemma 3.4.11 for even p. We omit the details here. □

Corollary 3.4.15. *Let the boundary of Ω be connected. The space $\mathbf{E}^p_{\mathcal{T},\mathrm{nc}}$ can be decomposed as the direct sum*

$$\mathbf{E}^p_{\mathcal{T},\mathrm{nc}} = \left(\bigoplus_{V\in\mathcal{V}} \mathbf{B}^p_{V,\mathrm{nc}}\right) \oplus \left(\bigoplus_{E\in\mathcal{E}} \mathbf{B}^p_{E,\mathrm{nc}}\right) \oplus \left(\bigoplus_{\tau\in\mathcal{T}} \mathbf{B}^p_{\tau,\mathrm{nc}}\right). \tag{3.4.42}$$

Proof. The proof results directly connecting the results given in Corollary 3.3.3, Theorem 3.4.13 and Remarks 3.4.8, 3.4.10 and 3.4.12. □

Corollary 3.4.16. *The edge- and vertex-oriented subspaces are constructed such that the following direct sums decompositions hold*

$$\mathbf{B}^p_{E,\mathrm{nc}} \oplus \bigoplus_{\tau\in\mathcal{T}_E} \mathbf{B}^p_{\tau,\mathrm{nc}} = \left\{\mathbf{e} \in \mathbf{E}^p_{\mathcal{T},\mathrm{nc}} \mid \operatorname{supp}\mathbf{e} \subset \omega_E\right\} \ \forall E \in \mathcal{E}, \tag{3.4.43}$$

$$\mathbf{B}^p_{V,\mathrm{nc}} \oplus \bigoplus_{E\in\mathcal{E}_V} \mathbf{B}^p_{E,\mathrm{nc}} \oplus \bigoplus_{\tau\in\mathcal{T}_V} \mathbf{B}^p_{\tau,\mathrm{nc}} = \left\{\mathbf{e} \in \mathbf{E}^p_{\mathcal{T},\mathrm{nc}} \mid \operatorname{supp}\mathbf{e} \subset \omega_V\right\} \ \forall V \in \mathcal{V}. \tag{3.4.44}$$

Proposition 3.4.17. *Let the boundary of Ω be connected. The lowest order non-conforming intrinsic finite elements are given by*

$$\mathbf{E}^0_{\mathcal{T},\mathrm{nc}} = \operatorname{span}\left\{\nabla_{\mathcal{T}} U^E_1 : E \in \mathcal{E}\right\},$$

where the functions U^E_1 are the standard non-conforming Crouzeix-Raviart basis functions (cf. [30]).

Proof. Choosing $p = 0$ and taking into account that $N^1 = \mathcal{V}$ we conclude from (3.4.40) that a basis for $\mathbf{E}^0_{\mathcal{T},\mathrm{nc}}$ is given by $\bigcup_{E \in \mathcal{T}} \{\nabla_{\mathcal{T}} U_1^E\}$.

To show the connection to the Crouzeix-Raviart basis functions, we consider an edge $E \in \mathcal{E}$ with neighboring triangles τ_1 and τ_2. From (3.4.26), we deduce that U_1^E is affine on each of the triangles τ_1, τ_2 with value 1 at the endpoints of E and value -1 at the vertices of τ_1, τ_2 that are opposite to E. Hence, U_1^E coincides with the standard Crouzeix-Raviart basis functions; see again [30]. □

3.5 Conclusions

In this chapter we developed a general method for constructing of finite element spaces from intrinsic conforming and non-conforming conditions. As a model problem we have considered the Poisson equation, but this approach is by no means limited to this model problem. Using theoretical conditions in the spirit of the second Strang lemma, we have derived conforming and non-conforming finite element spaces of arbitrary order for the fluxes. For these spaces, we also derived sets of local basis functions.

It turns out that the lowest order non-conforming space is spanned by the trianglewise gradients of the standard non-conforming Crouzeix-Raviart basis functions. In general, all polynomial non-conforming spaces are spanned by the gradients of standard *hp*-finite element basis functions *enriched* by some edge oriented non-conforming basis functions for even polynomial degree and by some triangle-supported non-conforming basis functions for odd polynomial degree. As a by-product, this methodology allowed us to recover the well-known non-conforming Crouzeix-Raviart element [30] (cf. Proposition 3.4.17). By using a similar but more technical reasoning, it can be shown that our intrinsic derivation of non-conforming finite elements also allows to recover the second order non-conforming Fortin-Soulie element [36, 40], the third order Crouzeix-Falk element [29], and the family of Gauss-Legendre elements [8], [7].

4

Intrinsic FEM for Elasticity Problems

4.1 Introduction

In this chapter we extend the intrinsic approach presented in Chapter 3 to the linearized elasticity problem. We start with the intrinsic formulation of the pure traction problem in linearized elasticity and construct an intrinsic piecewise polynomial conforming finite element space for the direct approximation of the strain tensor. In addition to the constitutive equation, our intrinsic approach can also provide an approximation for the stress tensor.
A FEM intrinsic approach in linearized elasticity was first introduced in [24]. The variational form of the pure traction problem of linearized elasticity is reformulated here in terms of the stress tensor $\mathbf{e}$ as a minimization problem over the space $\mathbb{E}(\Omega)$ of all functions from $\mathbb{L}_s^2(\Omega)$ satisfying the Saint Venant compatibility conditions (2.4.1). In [4] a similar intrinsic approach for the pure displacement traction problem is proposed. A $\operatorname{curl}\mathbf{curl}$ free finite element in planar elasticity has been developed in [25] and [26] by transforming the lowest order Nédélec finite elements ([46], [47]) in $\operatorname{curl}\mathbf{curl}$ free elements by adding an appropriate constraint in each interior vertex of the triangulation. In this way the compatibility conditions required in the intrinsic formulation are satisfied. In [17] an intrinsic Lagrangian approach is developed for the pure traction and pure displacement problem in three-dimensional linearized elasticity. There were also attempts for intrinsic approaches in nonlinear elasticity ([21]). In the nonlinear case the Green-Saint Venant tensor $\mathbf{E}(\mathbf{V}) = \left(\nabla\mathbf{v}^T + \nabla\mathbf{v} + \nabla\mathbf{v}^T\nabla\mathbf{v}\right)/2$ or equivalently the Cauchy-Green tensor $\mathbf{I} + 2\mathbf{E}(v)$ are the primary unknowns.
Following the general method introduced in Chapter 3, our approach provides a basis for the intrinsic finite element space based on a local characterization and on a decomposition of the space in a direct sum of triangle-, edge-, and vertex-oriented local subspaces. It turns out that these local conditions for a conforming discretization lead to an intrinsic finite element space spanned by the symmetric gradients of standard hp-finite element basis functions.
This chapter uses the notations introduced in the Chapter 2, Section 2.2. The chapter is organized as follows. In Section 4.2 we define the model problem. Section 4.3 contains the main results. In this section it is introduced and proved the local characterization of the conforming finite element space as well as the isomorphism which allows us to give the intrinsic formulation of the problem.

4.2 Model Problem

We consider the pure traction linearized elasticity problem of a homogeneous*, isotropic†, linearly elastic body, having the reference configuration $\overline{\Omega} \subset \mathbb{R}^2$, with Ω an open, bounded, connected, simply-connected domain in $\mathbb{R}^2$, with a Lipschitz continuous boundary Γ‡. The body is subject to applied forces of density $\mathbf{f}$ in its interior and $\mathbf{g} = \mathbf{0}$ on its boundary. The following assumptions are imposed on the Lamé moduli:

Assumption 4.2.1. *The Lamé coefficients satisfy the conditions* $0 < \mu < \infty$, $0 \leq \lambda < \infty$.

It is known (c.f. [24, 25, 26, 35, 10]) that the weak formulation of this problem consists in finding a displacement vector field $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$ such that

$$\int_\Omega \mathbf{A}\nabla_s \mathbf{u} : \nabla_s \mathbf{v} = L(\mathbf{v}), \text{ for all } \mathbf{v} \in \mathbf{H}_0^1(\Omega), \tag{4.2.1}$$

where $L(\mathbf{v}) := \displaystyle\int_\Omega \mathbf{f} \cdot \mathbf{v}$ and $\mathbf{Ae}$ is the elasticity tensor defined in (1.4.5).
The variational problem (4.2.1) is equivalent to the following minimization problem:
Find $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$ *such that*

$$J(\mathbf{u}) = \inf_{\mathbf{v} \in \mathbf{H}_0^1(\Omega)} J(\mathbf{v}), \tag{4.2.2}$$

where

$$J(\mathbf{v}) = \frac{1}{2}\int_\Omega \mathbf{A}\nabla_s \mathbf{v} : \nabla_s \mathbf{v} - L(\mathbf{v}).$$

It is well known that the variational equation (4.2.1) has a solution if and only if the compatibility condition $L(\mathbf{v}) = 0$ is satisfied for all $\mathbf{v} \in \mathbf{R}(\Omega)$, where $\mathbf{R}(\Omega)$ is the space of infinitesimal rigid displacement fields of Ω, defined in (1.4.10). In the case of a domain $\Omega \subset \mathbb{R}^2$, $\mathbf{R}(\Omega)$ is given by:

$$\mathbf{R}(\Omega) = \left\{ \mathbf{r} \in \mathbf{H}^1(\Omega); \mathbf{r} = \mathbf{a} + b \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix};\ \mathbf{a} \in \mathbb{R}^2,\ b \in \mathbb{R} \right\}. \tag{4.2.3}$$

Moreover, taking into account that $\mathbf{ker}\nabla_\mathbf{s} = \{0\}$ in $\mathbf{H}_0^1(\Omega)$, the solution of the variational equation (4.2.1) and equivalently of the minimization problem (4.2.2) is unique in $\mathbf{H}_0^1(\Omega)$.
The intrinsic method of solving the elasticity problem formulated above consists in finding directly the approximation of the linearized strain tensor field $\mathbf{e}(\mathbf{u}) = \nabla_s \mathbf{u}$ instead of finding the approximation of the displacement vector field $\mathbf{u}$.
Taking into account the considerations presented in Section 2.4, the intrinsic form of the variational equation (4.2.1) and the minimization problem (4.2.2) are based on the space $\mathbb{E}_2(\Omega)$ of symmetric matrix fields $\mathbf{e} \in \mathbb{L}_s^2(\Omega)$ satisfying the weak form of Saint Venant's compatibility conditions. These conditions assure, for any symmetric matrix field $\mathbf{e} \in \mathbb{L}_s^2(\Omega)$, the existence of a vector field $\mathbf{v} \in \mathbf{H}^1(\Omega)$ such that $\mathbf{e} = \nabla_s \mathbf{v}$. It is known [4, 25] that the weak version of the classical Saint Venant compatibility conditions on the tensor field $\mathbf{e} \in \mathbb{L}_s^2(\Omega)$ is represented by the single equation:

$$\partial_{11}e_{22} - 2\partial_{12}e_{12} + \partial_{22}e_{11} = 0 \text{ in } \mathbf{H}^{-2}(\Omega) \tag{4.2.4}$$

*For an homogeneous body the elasticity tensor $\mathbf{A}$, defined in section 1.4.2, is independent of $\mathbf{x}$ and the Lamé moduli are constants.

†An isotropic body has an invariant response under rotations and the expression of its elasticity tensor $\mathbf{Ae}$ is given by the relation (1.4.5).

‡A detailed description of the pure traction linearized elasticity problem and of the notations used is given in Section 1.4.2.

or in shorty curl $\mathbf{curl}(\mathbf{e}) = 0$ in $\mathbf{H}^{-2}(\Omega)$.
Consequently, the space $\mathbb{E}_2(\Omega)$ is defined as

$$\mathbb{E}_2(\Omega) = \left\{ \mathbf{e} \in \mathbb{L}_s^2(\Omega); \int_\Omega \mathbf{e} \,:\, \mathbf{CURL}(\mathbf{curl}(v)) = 0,\ \forall v \in H^2(\Omega) \right\}. \tag{4.2.5}$$

We recall below some well-know results that we need for the intrinsic approach*.

Theorem 4.2.2. *([24, 26]) Let Ω be an open, bounded, connected, simply-connected subset of $\mathbb{R}^2$ with Lipschitz continuous boundary and $\mathbf{e} \in \mathbb{E}_2(\Omega)$. Then there exists a vector field $\mathbf{v} \in \mathbf{H}^1(\Omega)$ such that $\mathbf{e} = \nabla_s \mathbf{v}$ in $L^2(\Omega)$ and all other solutions of the equation $\mathbf{e} = \nabla_s(\tilde{\mathbf{v}})$ are of the form $\tilde{\mathbf{v}} = \mathbf{v} + \mathbf{r}$ for some $\mathbf{r} \in \mathbf{R}(\Omega)$, where $\mathbf{R}(\Omega)$ is the space of infinitesimal rigid displacement fields of Ω.*

The proof is based on a H^{-2}-version of the classical Poincaré theorem, proved in [24], which requires the simply-connected property of Ω.

Theorem 4.2.3. *([26]) Let Ω be a bounded, connected, simply-connected subset of $\mathbb{R}^2$ with Lipschitz continuous boundary, $\mathbf{e} \in \mathbb{E}_2(\Omega)$ and $\mathbf{v}$ the unique element from $\mathbf{H}_0^1(\Omega)$ that satisfies $\nabla_s \mathbf{v} = \mathbf{e}$, then the operator $\mathfrak{F} : \mathbb{E}_2(\Omega) \to \mathbf{H}_0^1(\Omega)$, $\mathfrak{F}(\mathbf{e}) = \mathbf{v}$ is an isomorphism and its inverse operator is $\nabla_s : \mathbf{H}_0^1(\Omega) \to \mathbb{E}_2(\Omega)$.*

The isomorphism $\mathfrak{F}$ allows us to reformulate the variational problem (4.2.1) in the intrinsic approach as follows:
Find a matrix field $\mathbf{e} \in \mathbb{E}_2(\Omega)$ such that

$$\int_\Omega \mathbf{A}\mathbf{e} : \tilde{\mathbf{e}} = l(\tilde{\mathbf{e}}),\ \textit{for all } \tilde{\mathbf{e}} \in \mathbb{E}_2(\Omega),\ \textit{with } l = L \circ \mathfrak{F}. \tag{4.2.6}$$

Equivalently, the minimization problem (4.2.2) can be rewritten as:
Find $\mathbf{e} \in \mathbb{E}_2(\Omega)$ such that

$$J(\mathbf{e}) = \inf_{\tilde{\mathbf{e}} \in \mathbb{E}_2(\Omega)} J(\tilde{\mathbf{e}}),\ \textit{where } J(\tilde{\mathbf{e}}) = \frac{1}{2} \int_\Omega \mathbf{A}\mathbf{e} : \mathbf{e} - l(\mathbf{e}). \tag{4.2.7}$$

Thanks to the isomorphism $\mathfrak{F}$ the following theorem holds:

Theorem 4.2.4. *([24, 25, 26]) Let Ω be an open, bounded, connected, simply-connected subset of $\mathbb{R}^2$ with Lipschitz continuous boundary. The minimization problem defined in (4.2.7) has one and only one solution $\mathbf{e}$. Furthermore, $\mathbf{e} = \nabla_s \mathbf{u}$, where $\mathbf{u}$ is the unique solution from $\mathbf{H}_0^1(\Omega)$ of the classical variational formulation of the pure traction problem of linearized elasticity.*

Taking into account the dependence between the strain tensor ε and the stress tensor σ, given by the constitutive equation $\sigma = \mathbf{A}\varepsilon$ we can conclude that the minimization problem (4.2.7) also gives directly the stresses σ inside the elastic body. Therefore the intrinsic approach has direct applicability in many practical problems and in the next sections we will derive conforming finite element spaces for the direct approximation of $\mathbf{e}$ from (4.2.6).

*Similar results in $\mathbb{R}^3$ are formulated in Section 2.4.1.

4.3 Conforming Intrinsic Finite Element Spaces

In the following we consider only the case of two-dimensional, bounded, polygonal domains $\Omega \subset \mathbb{R}^2$ and simplicial triangulations. The triangulation $\mathcal{T}$ of Ω is regular in the sense of [23]. All notations and hypothesis about the triangulation's elements were introduced in Section 2.2.
For a given triangulation $\mathcal{T}$ we define the finite element spaces:

$$\begin{aligned}
S_{\mathcal{T}}^{p,m} &:= \left\{u \in H^{m+1}(\Omega) \mid \forall \tau \in \mathcal{T} : u|_{\overset{\circ}{\tau}} \in \mathcal{P}_p\right\},\\
\mathbf{S}_{\mathcal{T}}^{p,m} &:= S_{\mathcal{T}}^{p,m} \times S_{\mathcal{T}}^{p,m},\\
S_{\mathcal{T},0}^{p,m} &:= S_{\mathcal{T}}^{p,m} \cap H_0^1(\Omega),\\
\mathbf{S}_{\mathcal{T},0}^{p,m} &:= S_{\mathcal{T},0}^{p,m} \times S_{\mathcal{T},0}^{p,m},\\
\mathbb{S}_{\mathcal{T}}^{p} &:= \left\{\mathbf{e} \in \mathbb{L}_s^2(\Omega) \mid \mathbf{e}|_{\overset{\circ}{\tau}} \in \mathbb{P}_s^p\right\},
\end{aligned}$$

$$\mathbb{E}_{\mathcal{T}}^{p} := \left\{\mathbf{e} \in \mathbb{S}_{\mathcal{T}}^{p} \,\middle|\, \int_\Omega \mathbf{e} : \mathbf{CURL}(\mathbf{curl}(v)) = 0,\ \forall v \in H^2(\Omega)\right\}. \tag{4.3.1}$$

Note that (4.2.5) implies the inclusion $\mathbb{E}_{\mathcal{T}}^{p} \subset \mathbb{E}_2(\Omega)$ holds. The piecewise polynomial finite element space $\mathbb{E}_{\mathcal{T}}^{p}$ leads to the following conforming Galerkin discretization of (4.2.6):
Find a matrix field $\mathbf{e}_{\mathcal{T}} \in \mathbb{E}_{\mathcal{T}}^{p}$ such that

$$\int_\Omega \mathbf{A}\mathbf{e}_{\mathcal{T}} : \tilde{\mathbf{e}}_{\mathcal{T}} = l(\tilde{\mathbf{e}}_{\mathcal{T}}),\ \textit{for all}\ \tilde{\mathbf{e}}_{\mathcal{T}} \in \mathbb{E}_{\mathcal{T}}^{p}. \tag{4.3.2}$$

The goal of the next part of Section 4.3 is to derive a local basis for the space $\mathbb{E}_{\mathcal{T}}^{p}$.

4.3.1 Partial integration formulas

The aim of this subsection is to prove the integration formulas required for the local characterization of the finite element space $\mathbb{E}_{\mathcal{T}}^{p}$. Partial integration gives us*

$$\int_\Omega v \operatorname{curl} \mathbf{w} = \int_{\partial\Omega} v\,(\mathbf{n} \times \mathbf{w}) + \int_\Omega \mathbf{curl}\,(v) \cdot \mathbf{w}.$$

Furthermore, we get for $\mathbf{e} \in \mathbb{S}^2$

$$\int_\Omega \mathbf{w} \cdot \mathbf{curl}\,(\mathbf{e}) = \int_{\partial\Omega} \mathbf{w} \cdot (\mathbf{n} \times \mathbf{e}) + \int_\Omega \mathbf{CURL}\,(\mathbf{w}) : \mathbf{e}.$$

Hence, it follows

$$\begin{aligned}
\int_\Omega v \operatorname{curl} \mathbf{curl}\,(\mathbf{e}) &= \int_{\partial\Omega} v\,(\mathbf{n} \times \mathbf{curl}\,(\mathbf{e})) + \int_\Omega \mathbf{curl}\,(v) \cdot \mathbf{curl}\,(\mathbf{e})\\
&= \int_{\partial\Omega} v\,(\mathbf{n} \times \mathbf{curl}\,(\mathbf{e})) + \int_{\partial\Omega} \mathbf{curl}\,(v) \cdot (\mathbf{n} \times \mathbf{e})\\
&\quad + \int_\Omega \mathbf{CURL}\,(\mathbf{curl}\,(v)) : \mathbf{e}.
\end{aligned}$$

*Note that this formula can be written also in the form

$$\int_\Omega v\,(\nabla \times \mathbf{w}) = \int_{\partial\Omega} v\,(\mathbf{n} \times \mathbf{w}) + \int_\Omega (\mathbf{w} \times \nabla)\,v.$$

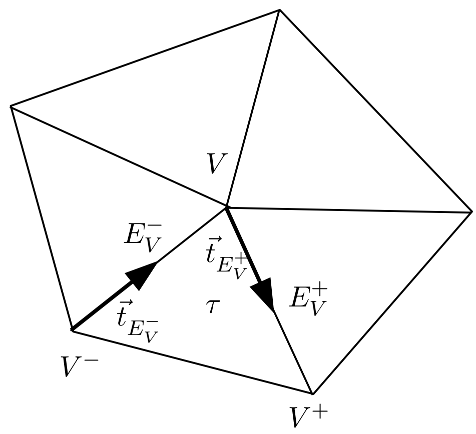

Figure 4.1: *Notations for partial integration on ω_V.*

Note that for $\mathbf{w} : \Omega \to \mathbb{R}^2$ and $\mathbf{e} : \Omega \to \mathbb{R}^{2\times 2}$ it holds

$$\mathbf{n} \times \mathbf{w} = -(\mathbf{t} \cdot \mathbf{w}) \quad \text{and} \quad \mathbf{n} \times \mathbf{e} = -\mathbf{et}.$$

Here, $\mathbf{t} = (n_2, -n_1)^\intercal$ is the unit tangential vector such that $\det[\mathbf{t}, \mathbf{n}] = 1$.
Hence,

$$\int_\Omega v \operatorname{curl} \mathbf{curl}(\mathbf{e}) = \int_\Omega \mathbf{CURL}(\mathbf{curl}(v)) : \mathbf{e} - \int_{\partial\Omega} v\,(\mathbf{t} \cdot \mathbf{curl}(\mathbf{e})) - \int_{\partial\Omega} \mathbf{curl}(v) \cdot (\mathbf{et})\,. \tag{4.3.3}$$

Let $\mathcal{T}$ be a finite element triangulation of a bounded polygonal domain $\Omega \subset \mathbb{R}^2$, $\mathcal{T}_V$ defined in (2.2.12) and v be a smooth function with $v|_{\partial\Omega} = 0$. For any triangle $\tau \in \mathcal{T}_V$ there are two edges E_V^- and E_V^+ which have V as an endpoint. The other end point in $E_V^\pm$ is denoted by $V^\pm$ and the length of an edge E is denoted by $|E|$. The numbering convention is such that $V = V^- + \left|E_V^-\right| \mathbf{t}_{E_V^-}$(cf. Figure 2.3).

Now we express $(\mathbf{curl}(v))|_{\partial\Omega}$ by

$$(\mathbf{curl}(v))|_{\partial\Omega} = \frac{\partial v}{\partial \mathbf{n}}\mathbf{t} - \frac{\partial v}{\partial \mathbf{t}}\mathbf{n},$$

so that

$$(\mathbf{curl}(v) \cdot (\mathbf{et}))|_{\partial\Omega} = \frac{\partial v}{\partial \mathbf{n}}(\mathbf{t} \cdot (\mathbf{et})) - \frac{\partial v}{\partial \mathbf{t}}(\mathbf{n} \cdot (\mathbf{et}))\,.$$

Then, partial integration along $\partial\Omega$ yields

$$\begin{aligned}
\int_{\partial\Omega} \mathbf{curl}(v) \cdot (\mathbf{et}) &= \int_{\partial\Omega} \left(\frac{\partial v}{\partial \mathbf{n}}(\mathbf{t} \cdot (\mathbf{et})) - \frac{\partial v}{\partial \mathbf{t}}(\mathbf{n} \cdot (\mathbf{et})) \right) \\
&= \int_{\partial\Omega} \left(\frac{\partial v}{\partial \mathbf{n}}(\mathbf{t} \cdot (\mathbf{et})) + v \frac{\partial (\mathbf{n} \cdot (\mathbf{et}))}{\partial \mathbf{t}} \right) \\
&\quad - \sum_V \left(v(V)\,\mathbf{n}_{E_V^-} \cdot \left(\mathbf{e}(V)\,\mathbf{t}_{E_V^-}\right) - v(V)\,\mathbf{n}_{E_V^+} \cdot \left(\mathbf{e}(V)\,\mathbf{t}_{E_V^+}\right)\right)
\end{aligned}$$

and we obtain

$$\begin{aligned}\int_{\Omega} \mathbf{e} : \mathbf{CURL}\left(\mathbf{curl}\left(v\right)\right) = & \int_{\Omega} v \operatorname{curl} \mathbf{curl}\left(\mathbf{e}\right) \\ & + \int_{\partial\Omega} \left(\left(\mathbf{t} \cdot \left(\mathbf{e}\mathbf{t}\right)\right) \frac{\partial v}{\partial \mathbf{n}} + \left(\mathbf{t} \cdot \mathbf{curl}\left(\mathbf{e}\right) + \frac{\partial \left(\mathbf{n} \cdot \left(\mathbf{e}\mathbf{t}\right)\right)}{\partial \mathbf{t}} \right) v \right) \\ & - \sum_{V} v\left(V\right) \left(\mathbf{n}_{E_V^-} \cdot \left(\mathbf{e}\left(V\right) \mathbf{t}_{E_V^-} \right) - \mathbf{n}_{E_V^+} \cdot \left(\mathbf{e}\left(V\right) \mathbf{t}_{E_V^+} \right) \right).\end{aligned} \tag{4.3.4}$$

4.3.2 Local Characterization of Conforming Intrinsic Finite Elements

For any inner edge E let $\mathbf{n}_E$ be an unit vector orthogonal to the edge E, having an arbitrary, but fixed orientation. The orientation for the boundary edges is such that $\mathbf{n}_E$ points into the exterior of Ω. Let $\mathbf{t}_E$ denote the unit vector oriented along E and satisfying the condition $\det\left[\mathbf{t}_E, \mathbf{n}_E\right] = 1$. For the inner edges $E \in \mathcal{E}$ we define:

1. The jump across E of the matrix field $\mathbf{e}$ by

$$[\mathbf{e}]_E : E \to \mathbb{R},\ [\mathbf{e}]_E(\mathbf{x}) := \lim_{\varepsilon \searrow 0} \left(\mathbf{e}(\mathbf{x} + \varepsilon \mathbf{n}_E) - \mathbf{e}(\mathbf{x} - \varepsilon \mathbf{n}_E)\right),\ \forall \mathbf{x} \in \overset{\circ}{E}. \tag{4.3.5}$$

2. The pointwise tangential jumps $\left[\mathbf{e}\mathbf{t}_E\right]_E : E \to \mathbb{R}, \forall \mathbf{x} \in \overset{\circ}{E}$, by

$$\left[\mathbf{e}\mathbf{t}_E\right]_E\left(\mathbf{x}\right) := \lim_{\varepsilon \searrow 0} \left(\mathbf{e}\left(\mathbf{x} + \varepsilon \mathbf{n}_E\right) \mathbf{t}_E - \mathbf{e}\left(\mathbf{x} - \varepsilon \mathbf{n}_E\right) \mathbf{t}_E\right). \tag{4.3.6}$$

For an edge $E \in \mathcal{E}_V$ we denote the other endpoint by B_E. We set

$$s_{V,E} := \operatorname{sign}((V - B_E) \cdot \mathbf{t}_E). \tag{4.3.7}$$

In Figure 4.2 we illustrate the definition given in (4.3.7): for the edge E_1 we have $s_{V,E_1} = 1$ and for E_2 it holds $s_{V,E_2} = -1$.

We define:

$$[\mathbf{e}]_V := \sum_{E \in \mathcal{E}_V} s_{V,E}([\mathbf{e}]_E(V)\mathbf{t}_E \cdot \mathbf{n}_E). \tag{4.3.8}$$

Lemma 4.3.1. *Let the boundary of Ω be connected. The space $\mathbb{E}^p_{\mathcal{T}}$ defined in (4.3.1) can be characterized by local conditions according to*

$$\begin{aligned}\mathbb{E}^p_{\mathcal{T}} \quad = \quad & \Big\{ \mathbf{e} \in \mathbb{S}^p_{\mathcal{T}} \mid \operatorname{curl}_{\mathcal{T}} \mathbf{curl}_{\mathcal{T}}(\mathbf{e}) = 0, \\ & [(\mathbf{curl}_{\mathcal{T}}(\mathbf{e}) \cdot \mathbf{t}_E)]_E + \left[\frac{\partial}{\partial \mathbf{t}_E}(\mathbf{e}\mathbf{t}_E \cdot \mathbf{n}_E)\right]_E = 0,\ \forall E \in \mathcal{E}, \\ & [(\mathbf{e}\mathbf{t}_E \cdot \mathbf{t}_E)]_E = 0,\ \forall E \in \mathcal{E}, \\ & [\mathbf{e}]_V = 0,\ \forall V \in \mathcal{V} \Big\}.\end{aligned} \tag{4.3.9}$$

Proof. We denote the right-hand side in (4.3.9) by $\tilde{\mathbb{E}}^p_{\mathcal{T}}$ and prove that $\mathbb{E}^p_{\mathcal{T}} = \tilde{\mathbb{E}}^p_{\mathcal{T}}$.
In the first three parts (Part a - Part c) of the proof we will show that $\mathbb{E}^p_{\mathcal{T}} \subset \tilde{\mathbb{E}}^p_{\mathcal{T}}$. Let $\mathbf{e} \in \mathbb{E}^p_{\mathcal{T}}$.

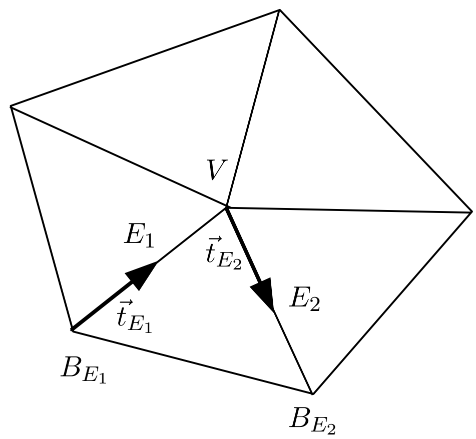

Figure 4.2: *Edges with different values for $s_{V,E}$.*

Part a: For $\tau \in \mathcal{T}$, let $v \in \mathcal{D}(\tau) := \{u \in C^\infty(\tau) | \operatorname{supp} u \subset\subset \tau\}$. Then, using (4.3.3) we obtain

$$\int_\tau (\operatorname{curl} \mathbf{curl}(\mathbf{e}))v = \int_\tau \mathbf{e} : \mathbf{CURL}(\mathbf{curl}(v)) = 0.$$

Since $\tau \in \mathcal{T}$ and $v \in \mathcal{D}(\tau)$ are arbitrary, we conclude that $\operatorname{curl}_\mathcal{T} \mathbf{curl}_\mathcal{T}(\mathbf{e}) = 0$ holds.

Part b: Let τ_1, τ_2 and $E \in \mathcal{E}$ be such that $E = \tau_1 \cap \tau_2$. Set $\omega_E := \tau_1 \cup \tau_2$. For $i = 1, 2$, let $\mathbf{n}_i$ denote the outer unit normal vector for τ_i, and, as a convention for the ordering of τ_1 and τ_2, we assume that $\mathbf{n}_1|_E = \mathbf{n}_E$. The tangential unit vector $\mathbf{t}_i$ is chosen such that $\det[\mathbf{t}_i, \mathbf{n}_i] = 1$. We choose $v \in \mathcal{D}(\omega_E)$ and define

$$I_E := \int_{\omega_E} \mathbf{e} : \mathbf{CURL}(\mathbf{curl}(v)). \tag{4.3.10}$$

We obtain (by using $\mathbf{n}_1|_E = -\mathbf{n}_2|_E$ and $\mathbf{t}_1|_E = -\mathbf{t}_2|_E$ and the fact that v vanishes in a neighbourhood of the boundary of ω_E) from (4.3.4) the relation

$$\begin{aligned} I_E &= \int_{\tau_1} \mathbf{e} : \mathbf{CURL}(\mathbf{curl}(v)) + \int_{\tau_2} \mathbf{e} : \mathbf{CURL}(\mathbf{curl}(v)) \\ &= \int_{\omega_E} v \operatorname{curl}_\mathcal{T} \mathbf{curl}_\mathcal{T}(\mathbf{e}) \\ &- \int_E \left([\mathbf{t}_E \cdot \mathbf{e}\mathbf{t}_E]_E \frac{\partial v}{\partial \mathbf{n}_E} + \left([\mathbf{t}_E \cdot \mathbf{curl}_\mathcal{T}(\mathbf{e})]_E + \left[\frac{\partial}{\partial \mathbf{t}_E} (\mathbf{n}_E \cdot \mathbf{e}\mathbf{t}_E) \right]_E \right) v \right). \end{aligned} \tag{4.3.11}$$

By comparing (4.3.10) and (4.3.11) we obtain by density that $I_E = 0$ is equivalent to

$$\begin{aligned} &\operatorname{curl} \mathbf{curl}(\mathbf{e}) = 0 \quad \text{in } \omega_E \backslash E \quad \text{and} \quad [\mathbf{t}_E \cdot \mathbf{e}\mathbf{t}_E]_E = 0 \quad \text{and} \\ &[\mathbf{t}_E \cdot \mathbf{curl}_\mathcal{T}(\mathbf{e})]_E + \left[\frac{\partial}{\partial \mathbf{t}_E} (\mathbf{n}_E \cdot \mathbf{e}\mathbf{t}_E) \right]_E = 0. \end{aligned}$$

Part c. Let us consider an inner vertex $V \in \mathcal{V}$ and the domain ω_V as in (2.2.12). For simplicity we assume that for all edges $E \in \mathcal{E}_V$ the tangential vector $\mathbf{t}_E$ points towards the vertex V (otherwise one has to use the signs $s_{V,E}$ as in (4.3.7)).

For $v \in \mathcal{D}(\omega_V) := \{u \in C^\infty(\tau) \mid \operatorname{supp} u \subset \omega_V\}$ we set

$$I_V := \int_{\omega_V} \mathbf{e} : \mathbf{CURL}\left(\mathbf{curl}\,(v)\right)$$

and obtain from (4.3.4)

$$\begin{aligned} I_V = &\int_{\omega_V} v \operatorname{curl}_{\mathcal{T}} \mathbf{curl}_{\mathcal{T}}(\mathbf{e}) \\ &- \sum_{E \in \mathcal{E}_V} \int_E \left([\mathbf{t}_E \cdot \mathbf{e}\mathbf{t}_E]_E \frac{\partial v}{\partial \mathbf{n}_E} + \left([\mathbf{t}_E \cdot \mathbf{curl}_{\mathcal{T}}(\mathbf{e})]_E + \left[\frac{\partial}{\partial \mathbf{t}_E} (\mathbf{n}_E \cdot \mathbf{e}\mathbf{t}_E) \right]_E \right) v \right) \\ &+ v(V) \sum_{E \in \mathcal{E}_V} [\mathbf{n}_E \cdot \mathbf{e}\mathbf{t}_E]_E (V). \end{aligned}$$

This implies again by density that $I_V = 0$ is equivalent to

$$\begin{aligned} \operatorname{curl} \mathbf{curl}(\mathbf{e}) = 0 \text{ in } \omega_V \backslash \left(\bigcup_{E \in \mathcal{E}_V} E \right) \qquad &\text{and} \quad [\mathbf{t}_E \cdot \mathbf{e}\mathbf{t}_E]_E = 0, \\ [\mathbf{t}_E \cdot \mathbf{curl}_{\mathcal{T}}(\mathbf{e})]_E + \left[\frac{\partial}{\partial \mathbf{t}_E} (\mathbf{n}_E \cdot \mathbf{e}\mathbf{t}_E) \right]_E = 0 \quad &\text{and} \qquad [\mathbf{e}]_V = 0 \end{aligned}$$

and from this the assertion follows.

Part d: In order to prove the reverse inclusion let $\mathbf{e} \in \tilde{\mathbb{E}}^p_{\mathcal{T}}$ and $v \in \mathcal{D}(\Omega)$. The following equality holds:

$$\begin{aligned} {}_{\mathcal{D}'(\Omega)}\langle \operatorname{curl} \mathbf{curl}(\mathbf{e}), v \rangle_{\mathcal{D}(\Omega)} = \int_\Omega \mathbf{e} : \mathbf{CURL}\left(\mathbf{curl}\,(v)\right) = &\sum_{\tau \in \mathcal{T}} \int_\tau v \operatorname{curl}_{\mathcal{T}} \mathbf{curl}_{\mathcal{T}}(\mathbf{e}) \\ &- \sum_{E \in \mathcal{E}} \int_E [\mathbf{t}_E \cdot \mathbf{e}\mathbf{t}_E]_E \frac{\partial v}{\partial \mathbf{n}_E} \\ &- \sum_{E \in \mathcal{E}} \int_E \left([\mathbf{t}_E \cdot \mathbf{curl}_{\mathcal{T}}(\mathbf{e})]_E + \left[\frac{\partial}{\partial \mathbf{t}_E} (\mathbf{n}_E \cdot \mathbf{e}\mathbf{t}_E) \right]_E \right) v \\ &+ \sum_{V \in \mathcal{V}} [e]_V \, v = 0 \end{aligned}$$

which shows that $\tilde{\mathbb{E}}^p_{\mathcal{T}} \subset \mathbb{E}^p_{\mathcal{T}}$.
Consequently $\mathbb{E}^p_{\mathcal{T}} = \tilde{\mathbb{E}}^p_{\mathcal{T}}$. □

4.3.3 Integration

Let:

$$\mathbb{P}^p_{\text{curlcurl}} := \{\mathbf{e} \in \mathbb{P}^p_s \ : \operatorname{curl} \mathbf{curl}\, \mathbf{e} = \mathbf{0}\} \tag{4.3.12}$$

and, for $\tau \in \mathcal{T}$, we write $\mathbb{P}^p_{\text{curlcurl}}(\tau) := \left\{ \mathbf{e}|_\tau : \mathbf{e} \in \mathbb{P}^p_{\text{curlcurl}} \right\}$ to indicate the domain of the functions explicitly.

Lemma 4.3.2. *For any $\tau \in \mathcal{T}$ and any $\mathbf{e} \in \mathbb{P}^p_{\text{curlcurl}}(\tau)$, it holds*

$$\emptyset \neq \left\{ \mathbf{v} \in \mathbf{H}^1(\tau) \mid \nabla_s \mathbf{v} = \mathbf{e} \right\} \subset \mathbf{P}_{p+1}(\tau). \tag{4.3.13}$$

Proof. Let $\tau \in \mathcal{T}$ and $\mathbf{e} \in \mathbb{P}^p_{\text{curlcurl}}(\tau)$. It is proved in [24] that there exists $\mathbf{v} \in \mathbf{H}^1(\tau)$, unique up to an infinitesimal rigid displacement field of τ such that $\mathbf{e} = \nabla_s \mathbf{v}$ and this proves the left-hand size in (4.3.13).
Consider $\mathbf{e} \in \mathbb{P}^p_{\text{curlcurl}}$ as a 2×2 symmetric matrix of global polynomials in $\mathbb{R}^2$. As a consequence of the H^{-1}-version of Poincaré's theorem ([24]), a similar proof as in [28] yields that $\mathcal{G}(\mathbf{e})$ given by

$$\mathcal{G}(\mathbf{e})(\mathbf{x}) := \int_{\gamma_\mathbf{x}} \mathbf{e}(\mathbf{y})\mathbf{dy} + \int_{\gamma_\mathbf{x}} (\mathbf{x}-\mathbf{y}) \times \mathbf{curl}(\mathbf{e}(\mathbf{y}))\,\mathbf{dy} \tag{4.3.14}$$

with $\gamma_\mathbf{x} : [0,1] \to \overline{O\mathbf{x}}$; $\gamma_\mathbf{x}(t) = t\mathbf{x}$, satisfies $\nabla_s \mathcal{G}(\mathbf{e}) = \mathbf{e}$. For $\boldsymbol{\nu} = (\nu_1, \nu_2)^T \in \mathbb{N}_0^2$, we employ the usual multiindex notation: $|\boldsymbol{\nu}| := \nu_1 + \nu_2$ and for a vector $\mathbf{w} = (w_1, w_2)^T$ we set $\mathbf{w}^{\boldsymbol{\nu}} = w_1^{\nu_1} w_2^{\nu_2}$. For $1 \le i \le 4$, let $\mathbf{b}_i$ denote the canonical basis in $\mathbb{R}^{2\times 2}$, i.e., $\mathbf{b}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, etc. For $\mathbf{w} = \mathbf{x}^{\boldsymbol{\nu}} \mathbf{b}_i$ we obtain

$$\mathcal{G}(\mathbf{w})(\mathbf{x}) = \frac{1}{|\boldsymbol{\nu}|+1} \mathbf{x}^{\boldsymbol{\nu}} (\mathbf{b}_i \mathbf{x}) + \frac{1}{|\boldsymbol{\nu}|(|\boldsymbol{\nu}|+1)} \left((\nabla \mathbf{x}^{\boldsymbol{\nu}}) \times \mathbf{b}_i \mathbf{x} \right) \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \in \mathbf{P}_{|\boldsymbol{\nu}|+1}.$$

An element $\mathbf{e} \in \mathbb{P}^p_{\text{curlcurl}}$ is a linear combination of terms of the form $\mathbf{x}^{\boldsymbol{\nu}} \mathbf{b}_i$ for $1 \le i \le 4$, $0 \le |\boldsymbol{\nu}| \le p$ and an element from $\mathbf{R}(\tau)$. From (4.2.3) we get $\mathbf{R}(\tau) \subset \mathbf{P}_1$ such that $\mathcal{G}(\mathbf{e}) + \mathbf{R}(\tau) \subset \mathbf{P}_{p+1}$ and the assertion follows. □

Based on Lemma 4.3.2 we define the local lifting $\mathfrak{f}^{\mathbf{r}}_\tau : \mathbb{P}^p_{\text{curlcurl}}(\tau) \to \mathbf{P}_{p+1}(\tau)$:

$$\mathfrak{f}^{\mathbf{r}}_\tau(\mathbf{e}) := \mathcal{G}(\mathbf{e}) + \mathbf{r}, \tag{4.3.15}$$

for $\tau \in \mathcal{T}$, $\mathbf{e} \in \mathbb{P}^p_{\text{curlcurl}}(\tau)$ and $\mathbf{r}$ an infinitesimal rigid displacement field of τ, $\mathbf{r} \in \mathbf{R}(\tau)$.
The set in (4.3.13) satisfies:

$$\left\{ \mathbf{v} \in \mathbf{H}^1(\tau) \mid \mathbf{e} \in \mathbb{P}^p_{\text{curlcurl}}(\tau), \nabla_s \mathbf{v} = \mathbf{e} \right\} = \{ \mathfrak{f}^{\mathbf{r}}_\tau(\mathbf{e}) \mid \mathbf{r} \in \mathbf{R}(\tau) \}. \tag{4.3.16}$$

Proposition 4.3.3. *Let the boundary of Ω be connected. From (4.3.16) we conclude that the operator $\nabla_s : \mathbf{S}^{p+1,0}_{\mathcal{T},0} \to \mathbb{E}^p_{\mathcal{T}}$ is an isomorphism with inverse $\mathfrak{F} : \mathbb{E}^p_{\mathcal{T}} \to \mathbf{S}^{p+1,0}_{\mathcal{T},0}$.*

Proof. From Theorem 4.2.3 and Lemma 4.3.2 we conclude that

$$\mathfrak{F}(\mathbb{E}^p_{\mathcal{T}}) \subset \mathbf{S}^{p+1,-1}_{\mathcal{T}}$$

holds. Since $\mathbb{E}^p_{\mathcal{T}} \subset \mathbb{E}_2(\Omega)$, the mapping properties of the lifting $\mathfrak{F}$ defined in Theorem 4.2.3 imply

$$\mathfrak{F}(\mathbb{E}^p_{\mathcal{T}}) \subset \mathfrak{F}(\mathbb{E}_2(\Omega)) = \mathbf{H}^1_0(\Omega).$$

Hence

$$\mathfrak{F}(\mathbb{E}^p_{\mathcal{T}}) \subset \mathbf{S}^{p+1,-1}_{\mathcal{T}} \cap \mathbf{H}^1_0(\Omega) = \mathbf{S}^{p+1,0}_{\mathcal{T},0}. \tag{4.3.17}$$

On the other hand, we have $\mathbf{S}^{p+1,0}_{\mathcal{T},0} \subset \mathbf{H}^1_0(\Omega)$. Using the isomorphism ∇_s from Theorem 4.2.3, the inclusion

$$\nabla_s(\mathbf{S}^{p+1,0}_{\mathcal{T},0}) \subset \mathbb{E}_2$$

holds. Furthermore, it is clear that

$$\nabla_s(\mathbf{S}^{p+1,0}_{\mathcal{T},0}) \subset \mathbb{S}^{p,-1}_{\mathcal{T}}.$$

Consequently

$$\nabla_s(\mathbf{S}^{p+1,0}_{\mathcal{T},0}) \subset \mathbb{S}^{p,-1}_{\mathcal{T}} \cap \mathbb{E}_2 = \mathbb{E}^p_{\mathcal{T}},$$

which implies

$$\mathbf{S}^{p+1,0}_{\mathcal{T},0} \subset \mathfrak{F}(\mathbb{E}^p_{\mathcal{T}}). \tag{4.3.18}$$

Together with the inclusion (4.3.17) this first implies $\mathbf{S}^{p+1,0}_{\mathcal{T},0} = \mathfrak{F}(\mathbb{E}^p_{\mathcal{T}})$ and this completes the proof. □

4.3.4 A local basis for Conforming Intrinsic Finite Elements

From Proposition 4.3.3 we deduce that a basis for the spaces $\mathbb{B}^p_V$, $\mathbb{B}^p_E$, $\mathbb{B}^p_\tau$ can easily be constructed by using the standard basis functions for hp-finite element spaces (cf. [53]). Let $\mathcal{N}^p$ be the set of knots defined in (2.2.16). The Lagrange basis for $\mathbf{S}^{p,0}_{\mathcal{T},0}$ can be indexed by the nodal points $N \in \mathcal{N}^p$ and is characterized by

$$\mathbf{b}^{\mathcal{T}}_{p,N} \in \mathbf{S}^{p,0}_{\mathcal{T},0} \quad \text{and} \quad \forall N' \in \mathcal{N}^p \quad \mathbf{b}^{\mathcal{T}}_{p,N}\left(N'\right) = \begin{cases} 1 & N = N', \\ 0 & N \neq N'. \end{cases} \tag{4.3.19}$$

We define the following subspaces of $\mathbb{E}^p_{\mathcal{T}}$:

$$\mathbb{B}^p_\tau = span\left\{\nabla_s \mathbf{b}^{\mathcal{T}}_{p+1,N} \mid N \in \overset{\circ}{\tau} \cap \mathcal{N}^{p+1}\right\} \text{ for all } \tau \in \mathcal{T}, \tag{4.3.20}$$

$$\mathbb{B}^p_E = span\left\{\nabla_s \mathbf{b}^{\mathcal{T}}_{p+1,N} \mid N \in \overset{\circ}{E} \cap \mathcal{N}^{p+1}\right\} \text{ for all } E \in \mathcal{E}, \tag{4.3.21}$$

$$\mathbb{B}^p_V = span\left\{\nabla_s \mathbf{b}^{\mathcal{T}}_{p+1,V}\right\} \text{ for all } V \in \mathcal{V}. \tag{4.3.22}$$

Proposition 4.3.4. *Let the boundary of Ω be connected. The space $\mathbb{E}^p_{\mathcal{T}}$ can be decomposed as the direct sum*

$$\mathbb{E}^p_{\mathcal{T}} = \left(\bigoplus_{V\in\mathcal{V}} \mathbb{B}^p_V\right) \oplus \left(\bigoplus_{E\in\mathcal{E}} \mathbb{B}^p_E\right) \oplus \left(\bigoplus_{\tau\in\mathcal{T}} \mathbb{B}^p_\tau\right). \tag{4.3.23}$$

Proof. Proposition 4.3.3 implies that $(\nabla_s \mathbf{b}^{\mathcal{T}}_{p+1,N})_{N\in\mathcal{N}^{p+1}}$ is a basis of $\mathbb{E}^p_{\mathcal{T}}$. The assertion follows simply by sorting these basis functions, according to whether they are associated with a single triangle, with two triangles with a side in common, and with triangles with a vertex in common. □

Corollary 4.3.5. *The subspaces defined in (4.3.20), (4.3.21), (4.3.22) are triangle-, edge-, and vertex-oriented local subspaces of $\mathbb{E}^p_{\mathcal{T}}$ and can be expressed as follows:*
The triangle-oriented subspace $\mathbb{B}^p_\tau$ is given by:

$$\mathbb{B}^p_\tau = \left\{\mathbf{e} \in \mathbb{E}^p_{\mathcal{T}} \mid \operatorname{supp} \mathbf{e} \subset \tau\right\}. \tag{4.3.24}$$

The edge-oriented subspace $\mathbb{B}^p_E$ satisfies the direct sum decomposition

$$\mathbb{B}^p_E \oplus \left(\bigoplus_{\tau\in\mathcal{T}_E} \mathbb{B}^p_\tau\right) = \left\{\mathbf{e} \in \mathbb{E}^p_{\mathcal{T}} \mid \operatorname{supp} \mathbf{e} \subset \omega_E\right\}. \tag{4.3.25}$$

The vertex-oriented subspace $\mathbb{B}^p_V$ is constructed such that the following direct sum decomposition holds

$$\mathbb{B}^p_V \oplus \left(\bigoplus_{E\in\mathcal{E}_V} \mathbb{B}^p_E\right) \oplus \left(\bigoplus_{\tau\in\mathcal{T}_V} B^p_\tau\right) = \left\{\mathbf{e} \in \mathbb{E}^p_{\mathcal{T}} \mid \operatorname{supp} \mathbf{e} \subset \omega_V\right\}. \tag{4.3.26}$$

Remark 4.3.6. There are no other conforming basis functions of the space $\mathbb{E}_{\mathcal{T}}^{p}$.

Remark 4.3.7. Proposition *4.3.4* and the definition of triangle-, edge-, and vertex-oriented local subspaces of $\mathbb{E}_{\mathcal{T}}^{p}$ shows that *(4.3.2)* is equivalent to the standard Galerkin finite element formulation:
Find $\mathbf{u}_{\mathcal{T}} \in \mathbf{S}_{\mathcal{T},0}^{p+1,0}$ *such that*

$$\int_{\Omega} \mathbf{A}\nabla_s \mathbf{u}_{\mathcal{T}} : \nabla_s \tilde{\mathbf{u}}_{\mathcal{T}} = l(\tilde{\mathbf{u}}_{\mathcal{T}}), \textit{ for all } \tilde{\mathbf{u}}_{\mathcal{T}} \in \mathbf{S}_{\mathcal{T},0}^{p+1,0}$$

via $\mathbf{e}_{\mathcal{T}} = \nabla_s \mathbf{u}_{\mathcal{T}}$.

4.4 Conclusions

In this chapter we applied the general method developed in Chapter 3 to obtain intrinsic conforming piecewise finite element spaces of arbitrary degree p for the pure traction problem. Our method provides a direct approximation of the strain tensor and, by using the constitutive equation, a direct approximation for the stress tensor is also obtained. The main idea is to construct a local characterization of the intrinsic finite element space. We have established an isomorphism which allows the construction of a local basis for the conforming intrinsic finite elements and the decomposition of the intrinsic finite element space into a direct sum of triangle-, edge- and vertex-oriented piecewise polynomial subspaces.

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